Asymptotic unitary equivalence in C^*-algebras

Huaxin Lin and Zhuang Niu

Abstract

Let $C = C(X)$ be the unital C*-algebra of all continuous functions on a finite CW complex X and let A be a unital simple C*-algebra with tracial rank at most one. We show that two unital monomorphisms $\varphi, \psi : C \to A$ are asymptotically unitarily equivalent, i.e., there exists a continuous path of unitaries $\{u_t : t \in [0,1)\} \subset A$ such that

$$\lim_{t\to 1} u_t^*\varphi(f)u_t = \psi(f) \text{ for all } f \in C(X),$$

if and only if

$$\begin{aligned} [\varphi] &= [\psi] \text{ in } KK(C,A), \\ \tau\circ\varphi &= \tau\circ\psi \text{ for all } \tau \in T(A), \text{ and} \\ \varphi^\dagger &= \psi^\dagger, \end{aligned}$$

where $T(A)$ is the simplex of tracial states of A and $\varphi^\dagger, \psi^\dagger : U(M_\infty(C))/DU(M_\infty(C)) \to U(M_\infty(A))/DU(M_\infty(A))$ are induced homomorphisms and where $U(M_\infty(A))$ and $U(M_\infty(C))$ are groups of union of unitary groups of $M_k(A)$ and $M_k(C)$ for all integer $k \geq 1$, $DU(M_\infty(A))$ and $DU(M_\infty(C))$ are commutator subgroups of $U(M_\infty(A))$ and $U(M_\infty(C))$, respectively. We actually prove a more general result for the case that C is any general unital AH-algebra.

1 Introduction

In the study of topology, it is fundamentally important to study continuous maps between topological spaces. In the study of C*-algebras, or sometime called the non-commutative topological space, it is essential to study homomorphisms from one C*-algebra to another.

One of the central problems in classification of amenable C*-algebras is to determine how certain equivalence classes of homomorphisms between C*-algebras can be determined by their K-theoretical invariants. In this note, we will study the unital monomorphisms from a unital commutative C*-algebra C, or, more general, arbitrary unital AH-algebras, to a simple C*-algebra A with finite tracial rank (see 2.6 below) and consider the question when two given unital monomorphisms $\varphi, \psi : C \to A$ are asymptotically unitarily equivalent, that is, when does there exist a continuous path of unitaries $\{u_t : t \in [0,1)\} \subset A$ such that

$$\lim_{t\to 1} u_t^*\varphi(f)u_t = \psi(f) \text{ for all } f \in C.$$

If one considers approximately unitary equivalence (recall that the maps φ and ψ are approximately unitarily equivalent if there exists a sequence of unitaies $\{u_n\} \subset A$ such that $\lim_{n\to\infty} u_n^*\varphi(f)u_n = \psi(f)$ for all $f \in C$), there are already several results recently:

Let C be a unital AH-algebra and let A be a unital simple C*-algebra with tracial rank zero. It has been shown in [6] by the first author that φ and ψ are approximately unitarily equivalent if and only if

$$[\varphi] = [\psi] \text{ in } \mathrm{KL(C,A)} \text{ and } \tau\circ\varphi = \tau\circ\psi \text{ for all } \tau \in \mathrm{T(A)}.$$

And in [17], Ng and Winter showed that the result above still holds if $C = C(X)$ with X a second countable, path connected, compact metric space and A is any simple unital separable nuclear C*-algebra which is real rank zero and $\mathcal{Z}$-stable, where $\mathcal{Z}$ is the Jiang-Su algebra.

Beyond the real rank zero case, in a more recent paper ([14]), it was shown that, if A is a unital simple C*-algebra with tracial rank at most one, then φ and ψ are approximately unitarily equivalent if and only if

$$\begin{aligned} [\varphi] &= [\psi] \text{ in } KL(C, A), \\ \tau \circ \varphi &= \tau \circ \psi \text{ for all } \tau \in T(A) \text{ and } \varphi^{\ddagger} = \psi^{\ddagger}, \end{aligned} \tag{e 1.1}$$

where $\varphi^{\ddagger}$, $\psi^{\ddagger}$: $U(M_{\infty}(C)/\overline{DU(M_{\infty}(C))} \to U(M_{\infty}(A))/\overline{DU(M_{\infty}(A))}$ are induced homomorphisms and $DU(M_{\infty}(C))$ and $DU(M_{\infty}(A))$ are commutator subgroups of $\cup_{k=1}^{\infty} U(M_k(C))$ and $\cup_{k=1}^{\infty} U(M_k(A))$, respectively.

These results play important roles in the recent progress of the Elliott program of the classification of amenable C*-algebras. It is natural to ask whether approximate unitary equivalence is the same as asymptotic unitary equivalence. It turns out, from a result of Kishimoto and Kumjian ([4]), that, in general, asymptotic unitary equivalence is different from approximate unitary equivalence. In particular, they studied the case that both A and C are unital simple $A\mathbb{T}$-algebras of real rank zero.

Then, in [10], the following criterion for asymptotical unitarily equivalent was developed for any unital AH-algebra C and any simple C*-algebra A with tracial rank zero: Suppose that $\varphi, \psi : C \to A$ are two unital monomorphisms. Then φ and ψ are asymptotically unitarily equivalent if and only if

$$[\varphi] = [\psi] \text{ in } KK(C, A),$$

$$\tau \circ \varphi = \tau \circ \psi \text{ for all } \tau \in T(A), \tag{e 1.2}$$

$$\overline{R_{\varphi,\psi}} = 0, \tag{e 1.3}$$

where $R_{\varphi,\psi}$ is the rotation map, which will be defined in 2.8.

It worth to point out that one application of this result is to the study of Voiculescu's AF-embedding problem: Let Ω be a compact metric space and let G be a finitely generated abelian group. Suppose that Λ is a G action on X. Then the above mentioned result can be used to prove that $C(\Omega) \rtimes_{\Lambda} G$ can be embedded into a unital simple AF-algebra if and only if Ω has a faithful Λ-invariant Borel probability measure.

There are other applications. With a method developed by Winter ([21]), the above mentioned asymptotic unitary equivalence result was also used to give an important advance in the Elliott program (see [21], [7] and [13]) for the C*-algebras which might be projectionless. An even further advance was made which allows the class of unital separable amenable simple C*-algebras classified by the conventional Elliott invariant to include C*-algebras which are so-called rationally finite tracial rank and their K_0-groups may not have the Riesz interpolation property. The technical key of this advance was the following asymptotic unitary equivalence theorem.

Theorem 1.1 (Theorem 7.2, [12]). *Let C be a unital simple AH-algebra of slow dimension growth and let A be any unital simple C*-algebra with tracial rank at most one. Suppose that $\varphi, \psi : C \to A$ are two unital monomorphisms. Then φ and ψ are asymptotically unitarily equivalent if and only if*

$$\begin{aligned} [\varphi] = [\psi] &\text{ in } KK(C, A), \\ \tau \circ \varphi = \tau \circ \psi &\textit{ for all } \tau \in T(A), \\ \varphi^{\ddagger} = \psi^{\ddagger}, &\text{ and } \overline{R_{\varphi,\psi}} = 0. \end{aligned}$$

However, while in [12], the theorem above also was proved for certain non-simple AH-algebras, it only includes those unital AH-algebras whose K-theory behave as low dimensional topological spaces. In this paper we will generalize the theorem above so that it will apply to all unital AH-algebras (with no restriction on dimension growth). In particular, it holds for $C = C(X)$ for any compact metric space X.

Moreover, in the case that $K_1(C)$ is finitely generated, we also find that the invariant could be simplified. In fact, in Theorem 4.8 below, the conditions that $\overline{R_{\varphi,\psi}} = 0$ and $\varphi^{\ddagger} = \psi^{\ddagger}$ can be simplified to the condition that $\varphi^{\dagger} = \psi^{\dagger}$, i.e., φ and ψ induce the same homomorphisms on $\cup_{k=1}^{\infty} U(M_k(C))/DU(M_{\infty}(C))$. However, we also point out that, in general, this simplification is not possible. A specific example will be presented.

Acknowledgement Most part of this work was done when both authors were in East China Normal University during 2012 Operator Algebra Program. The research of Z. N. is supported by an NSERC Discovery grant.

2 Preliminaries

2.1. Let A be a unital stably finite C*-algebra. Denote by $\mathrm{T}(A)$ the simplex of tracial states of A and denote by $\mathrm{Aff}(\mathrm{T}(A))$ the space of all real affine continuous functions on $\mathrm{T}(A)$. Suppose that $\tau \in \mathrm{T}(A)$ is a tracial state. We will also denote by τ the trace $\tau \otimes \mathrm{Tr}$ on $\mathrm{M}_k(A) = A \otimes \mathrm{M}_k(\mathbb{C})$ (for every integer $k \geq 1$), where Tr is the standard trace on $\mathrm{M}_k(\mathbb{C})$.

Denote by $\mathrm{M}_{\infty}(A)$ the set $\displaystyle\bigcup_{k=1}^{\infty} \mathrm{M}_k(A)$, where $\mathrm{M}_k(A)$ is regarded as a C*-subalgebra of $\mathrm{M}_{k+1}(A)$ by the embedding $\mathrm{M}_k(A) \ni a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_{k+1}(A)$.

For any projection $p \in \mathrm{M}_{\infty}(A)$, the evaluation $\tau \mapsto \tau(p)$ defines a positive affine function on $\mathrm{T}(A)$. This induces a canonical positive homomorphism $\rho_A : K_0(A) \to \mathrm{Aff}(\mathrm{T}(A))$.

Denote by $\mathrm{S}(A) := C_0((0,1)) \otimes A$ the suspension of A, denote by $\mathrm{U}(A)$ the unitary group of A, and denote by $\mathrm{U}(A)_0$ the connected component of $\mathrm{U}(A)$ containing the identity.

Let C be another unital C*-algebra and let $\varphi : C \to A$ be a unital *-homomorphism. Denote by $\varphi_{\mathrm{T}} : \mathrm{T}(A) \to \mathrm{T}(C)$ the continuous affine map induced by φ, i.e.,

$$\varphi_{\mathrm{T}}(\tau)(c) = \tau \circ \varphi(c)$$

for all $c \in C$ and $\tau \in \mathrm{T}(A)$. Denote by $\varphi_{\sharp} : \mathrm{Aff}(\mathrm{T}(C)) \to \mathrm{Aff}(\mathrm{T}(A))$ the map defined by

$$\varphi_{\sharp}(f)(\tau) = f(\varphi_{\mathrm{T}}(\tau))$$

for all $\tau \in \mathrm{T}(A)$.

Definition 2.2. Let A be a unital C*-algebra. Denote by $DU(A)$ the subgroup of generated by the commutators of $U(A)$ and denote by $CU(A)$ the closure of $DU(A)$. If $u \in U(A)$, its image in the quotient $U(A)/CU(A)$ will be denoted by $\overline{u}$.

Let B be another unital C*-algebra and let $\varphi : A \to B$ be a unital homomorphism. It is clear that φ maps $CU(A)$ into $CU(B)$. Let $\varphi^{\ddagger}$ denote the induced homomorphism from $U(A)/CU(A)$ into $U(B)/CU(B)$. It is also clear that φ maps $DU(A)$ into $DU(B)$. Denote by $\varphi^{\dagger} : U(A)/DU(A) \to U(B)/DU(B)$ the homomorphism induced by φ.

Let $n \geq 1$ be any integer. Denote by $U_n(A)$ the unitary group of $M_n(A)$, and denote by $DU_n(A)$ and $CU_n(A)$ the commutator subgroup of $U_n(A)$ and its closure, respectively. Regard $U_n(A)$ as a subgroup of $U_{n+1}(A)$ via the embedding $\mathrm{U}_k(A) \ni u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{U}_{k+1}(A)$,and denote by $U_{\infty}(A)$ the union of all $U_n(A)$.

Consider the union $CU_\infty(A) := \bigcup_n CU_n(A)$. It is then a normal subgroup of $U_\infty(A)$, and the quotient $U(A)_\infty/CU_\infty(A)$ is in fact isomorphic to the inductive limit of $U_n(A)/CU_n(A)$ (as abelian groups). Similarly, $DU_\infty(A) := \bigcup_n DU(A)_n$ is a normal subgroup of $U_\infty(A)$. We will use $\varphi^\ddagger$ for the homomorphism induced by φ from $U_\infty(A)/CU_\infty(A)$ into $U_\infty(B)/CU_\infty(B)$, and we will use $\varphi^\dagger$ for the homomorphism induced by φ from $U_\infty(A)/DU_\infty(A)$.

Remark 2.3. By Corollary 3.5 of [11], if A has tracial rank at most one (see 2.6 below), the map natural map

$$U(A)/CU(A) \to U(M_n(A))/CU(M_n(A))$$

is an isomorphism for any integer $n \geq 1$.

Definition 2.4. Let A be a unital C*-algebra, and let $u \in U(A)_0$. Let $u(t) \in C([0,1], A)$ be a piecewise-smooth path of unitaries such that $u(0) = u$ and $u(1) = 1$. Then the de la Harpe–Skandalis determinant of $u(t)$ is defined by

$$\mathrm{Det}(u(t))(\tau) = \frac{1}{2\pi i}\int_0^1 \tau(\frac{du(t)}{dt}u(t)^*)dt \text{ for all } \tau \in T(A),$$

which induces a homomorphism

$$\overline{\mathrm{Det}} : U(A)_0 \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}.$$

The determinant $\overline{\mathrm{Det}}$ can be extended to a map from $U_\infty(A)_0$ into $\mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$. It is easy to see that the determinant vanishes on the closure of commutator subgroup of $U_\infty(A)$. In fact, by 3.1 of [20], the closure of the commutator subgroup is exactly the kernel of this map, that is, it induces an isomorphism $\overline{\mathrm{Det}} : U_\infty(A)_0/CU_\infty(A) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$. Moreover, by ([20]), one has the following short exact sequence

$$0 \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))} \to U_\infty(A)/CU_\infty(A) \stackrel{\Pi}{\to} K_1(A) \to 0 \tag{e 2.1}$$

which splits (where the embedding of $\mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ induced by $(\overline{\mathrm{Det}})^{-1}$). We will fix a splitting map $s_1 : K_1(A) \to U_\infty(A)/CU_\infty(A)$. The notation Π and s_1 will be used late without further warning. For each $\bar{u} \in s_1(K_1(A))$, select and fix one element $u_c \in \bigcup_{n=1}^\infty M_n(A)$ such that $\overline{u_c} = \bar{u}$. Denote this set by $U_c(A)$. Moreover, in the case that A is unital, simple and $TR(A) \leq 1$ (see 2.6 below), one has that $U(A)/U_0(A)$ to $K_1(A)$ is an isomorphism and $\overline{\mathrm{Det}} : U_0(A)/CU(A) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ is also an isomorphism. Then one has

$$0 \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))} \to U(A)/CU(A) \to K_1(A) \to 0. \tag{e 2.2}$$

Definition 2.5. Let A be a unital C*-algebra and let C be a separable C*-algebra which satisfies the Universal Coefficient Theorem. Recall that $KL(C,A)$ is the quotient of $KK(C,A)$ modulo pure extensions. By a result of Dădărlat and Loring in [2], one has

$$KL(C,A) = \mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(A)), \tag{e 2.3}$$

where

$$\underline{K}(B) = (K_0(B) \oplus K_1(B)) \oplus (\bigoplus_{n=2}^\infty (K_0(B, \mathbb{Z}/n\mathbb{Z}) \oplus K_1(B, \mathbb{Z}/n\mathbb{Z})))$$

for any C*-algebra B. Then, in the rest of the paper, we will identify $KL(C,A)$ with $\mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(A))$. Let $\kappa \in KL(C,A)$. Denote by $\kappa_i : K_i(C) \to K_i(A)$ the homomorphism given by κ with $i = 0, 1$.

Definition 2.6. Let $k \geq 0$ be an integer. A unital simple C*-algebra A has tracial rank at most k, denoted by $\mathrm{TR}(A) \leq k$, if for any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$, and nonzero $a \in A^+$, there exist a nonzero projection $p \in A$ and a C*-subalgebra $I \cong \bigoplus_{i=1}^m C(X_i) \otimes M_{r(i)}$ with $1_I = p$ for some finite CW-complexes X_i with dimension at most k such that

(1) $\|xp - xp\| < \epsilon$ for any $x \in \mathcal{F}$,

(2) for any $x \in \mathcal{F}$, there is $x' \in I$ such that $\|pxp - x'\| \leq \epsilon$, and

(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

Moreover, if the C*-subalgebra I above can be chosen to be a finite dimensional C*-algebra, then A is said to have tracial rank zero, and in such case, we write $\mathrm{TR}(A) = 0$. It is a theorem of Guihua Gong [3] that every unital simple AH-algebra with no dimension growth has tracial rank at most one. It has been proved in [12] that every $\mathcal{Z}$-stable unital simple AH-algebra has tracial rank at most one. It is shown recently ([15]) that if a unital separable simple C*-algebra A satisfying the UCT has $TR(A) \leq k$, then $TR(A) \leq 1$.

Definition 2.7. Let A and B be two unital C*-algebras, and let ψ and φ be two unital monomorphisms from B to A. Then the mapping torus $M_{\varphi,\psi}$ is the C*-algebra defined by

$$M_{\varphi,\psi} := \{f \in \mathrm{C}([0,1], A);\ f(0) = \varphi(b) \text{ and } f(1) = \psi(b) \text{ for some } b \in B\}.$$

For any $\psi, \varphi \in \mathrm{Hom}(B, A)$, denoting by π_0 the evaluation of $M_{\varphi,\psi}$ at 0, we have the short exact sequence

$$0 \to \mathrm{S}(A) \to M_{\varphi,\psi} \to^{\pi_0} B \to 0.$$

If $\varphi_{*i} = \psi_{*i}$ $(i = 0, 1)$, then the corresponding six-term exact sequence breaks down to the following two extensions:

$$\eta_i(M_{\varphi,\psi}):\ 0 \to K_{i+1}(A) \to K_i(M_{\varphi,\psi}) \to K_i(B) \to 0\ \ (i = 0, 1).$$

2.8. Suppose that, in addition,

$$\tau \circ \varphi = \tau \circ \psi \ \text{ for all } \ \tau \in \mathrm{T}(A). \tag{e 2.4}$$

For any continuous piecewise smooth path of unitaries $u(t) \in M_{\varphi,\psi}$, consider the path of unitaries $w(t) = u^*(0)u(t)$ in A. Then it is a continuous and piecewise smooth path with $w(0) = 1$ and $w(1) = u^*(0)u(1)$. Denote by $R_{\varphi,\psi}(u) = \mathrm{Det}(w)$ the determinant of $w(t)$. It is clear with the assumption of (e 2.4) that $R_{\varphi,\psi}(u)$ depends only on the homotopy class of $u(t)$. Therefore, it induces a homomorphism, denoted by $R_{\varphi,\psi}$, from $K_1(M_{\varphi,\psi})$ to $\mathrm{Aff}(\mathrm{T}(A))$. One has the following lemma.

Lemma 2.9 (3.3 of [9], also see [4])**.** *When (e 2.4) holds, the following diagram commutes:*

$$\begin{array}{ccc} K_0(A) & \xrightarrow{[\imath]_1} & K_1(M_{\varphi,\psi}) \\ & \rho_A \searrow \qquad \swarrow R_{\varphi,\psi} & \\ & \mathrm{Aff}(\mathrm{T}(A)) & \end{array}$$

Definition 2.10. Fix two unital C*-algebras A and B with $\mathrm{TR}(A) \neq \emptyset$. Define $\mathcal{R}_0$ to be the subset of $\mathrm{Hom}(K_1(B), \mathrm{Aff}(T(A)))$ consisting of those homomorphisms $h \in \mathrm{Hom}(K_1(B), \mathrm{Aff}(T(A)))$ for which there exists a homomorphism $d : K_1(B) \to K_0(A)$ such that

$$h = \rho_A \circ d.$$

It is clear that $\mathcal{R}_0$ is a subgroup of $\mathrm{Hom}(K_1(B), \mathrm{Aff}(T(A)))$.

2.11. If $[\varphi] = [\psi]$ in $KK(B, A)$, then the exact sequences $\eta_i(M_{\varphi,\psi})$ $(i = 0, 1)$ split. In particular, there is a lifting $\theta : K_1(B) \to K_1(M_{\varphi,\psi})$. Consider the map

$$R_{\varphi,\psi} \circ \theta : K_1(B) \to \mathrm{Aff}(T(A)).$$

If a different lifting θ' is chosen, then, $\theta - \theta'$ maps $K_1(B)$ into $K_0(A)$. Therefore

$$R_{\varphi,\psi} \circ \theta - R_{\varphi,\psi} \circ \theta' \in \mathcal{R}_0.$$

Then define

$$\overline{R}_{\varphi,\psi} = [R_{\varphi,\psi} \circ \theta] \in \mathrm{Hom}(K_1(B), \mathrm{Aff}(T(A)))/\mathcal{R}_0.$$

See 3.4 of [12] for more details.

3 A basic homotopy lemma

The following is taken from Lemma 2.8 of [11].

Lemma 3.1. *Let C be a unital nuclear C*-algebra. Let $\mathcal{F} \subseteq C$ be a finite subset, $N \in \mathbb{N}$, and $\epsilon > 0$. There then exist a finite subset $\mathcal{G} \subseteq C$ and $\delta > 0$ such that for any unital C*-algebra A, any unitary $u \in A$ and any unital homomorphism $\varphi : C \to A$ with*

$$\|[\varphi(c), u]\| < \delta, \quad \forall c \in \mathcal{G},$$

there is a unital completely positive linear map $L : C \otimes \mathrm{C}(\mathbb{T}) \to A$ such that

$$\|L(f \otimes z^n) - \varphi(f)u^n\| < \epsilon, \quad \forall f \in \mathcal{F},\ -N \leq n \leq N.$$

Let X be a metric space. In the rest of the paper, we fix the metric on $X \times \mathbb{T}$ to be

$$\mathrm{dist}((x, t), (y, s)) = \sqrt{\mathrm{dist}(x, y)^2 + \mathrm{dist}(t, s)^2}, \quad \forall x, y \in X,\ s, t \in \mathbb{T}.$$

Definition 3.2 (5.2 of [8])**.** Recall that a unital simple C*-algebra A is said to be tracially approximately divisible if for any finite subset $\mathcal{F} \subseteq A$, any $\epsilon > 0$, any natural number N, and any $a \in A^+$, there is a C*-subalgebra $B \subseteq A$ with $B \cong M_k(\mathbb{C})$ for some $k \geq N$ such that if $p = 1_B$, then

(1) $\mathcal{F} \subseteq_\epsilon B' \cap A$, and

(2) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$,

where $B' \cap A$ is the relative commutant of B in A.

Remark 3.3. The definition above is slightly different—but equivalent—to the original definition in [8], in which the first condition is replaced by

(1') $\|cf - fc\| < \epsilon$ for any $f \in \mathcal{F}$ and any c in the unit ball of B.

Indeed, as in [1], for any finite dimensional C*-algebra $B \subseteq A$, one considers the conditional expectation

$$\mathbb{E}_B : A \ni a \mapsto \int_{U(B)} u^* a u d\mu,$$

where μ is the Haar measure on the unitary group $U(B)$. It is clear that $\mathbb{E}_B(a)$ commutes with B. Now, if $f \in A$ satisfies $\|fc - cf\| < \epsilon$ for any c in the unit ball of B, one has that

$$\|\mathbb{E}_B(f) - f\| < \epsilon.$$

In particular, this implies that $f \in_\epsilon B' \cap A$, and shows that the two definitions of tracially approximate divisibility are equivalent.

Similar to [11], for any nondecreasing function $\Delta : (0,1) \to (0,1)$ with $\lim_{t\to 0}\Delta(t) = 0$, define

$$\Delta_{00}(t) = \Delta(\frac{1}{2^{n+1}}), \quad \text{if } t \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}),$$

and

$$\Delta_0(t) = \frac{\sqrt{2}}{48}\Delta_0(t\sqrt{2}/6)t$$

Then Δ_{00} and Δ_0 are also nondecreasing and satisfy $\lim_{t\to 0}\Delta_{00}(t) = 0$, $\lim_{t\to 0}\Delta_0(t) = 0$.

Definition 3.4. Let X be a compact metric space and $P \in M_r(C(X))$ be a projection, where $r \geq 1$ is an integer. Put $C = PM_r(C(X))P$. Suppose $\tau \in T(C)$. It is known that there exists a probability measure μ_τ on X such that

$$\tau(f) = \int_X t_x(f(x))d\mu_\tau(x),$$

where t_x is the normalized trace on $P(x)M_rP(x)$ for all $x \in X$.

Remark 3.5. Regard $C(X)$ as the center of $C = PM_r(C(X))P$, and denote by $\iota : C(X) \to C$ the embedding. Then the measure μ_τ is in fact induced by the trace $\tau \circ \iota$ on $C(X)$.

Remark 3.6. The C*-algebra $(PM_r(C(X))P) \otimes C(\mathbb{T})$ is isomorphic to the homogeneous C*-algebra $\tilde{P}M_r(C(X\times\mathbb{T}))\tilde{P}$ with the projection $\tilde{P}$ given by $\tilde{P}(x,z) = P(x)$. Hence there is a natural embedding of $C(X\times\mathbb{T})$ into $(PM_r(C(X))P)\otimes C(\mathbb{T})$ as the center.

Lemma 3.7. *Let $C = PM_r(C(X))P$ for some compact metrizable space X, and let $\Delta : (0,1) \to (0,1)$ be a non-decreasing function and $\eta > 0$ such that*

$$\mu_{\tau\circ\varphi}(O_a) > \Delta(a) \text{ for all } \tau \in T(A)$$

and for any open ball O_a of X with radius $a > \eta$

Let $\mathcal{F} \subseteq C$, $\mathcal{G}' \subseteq C \otimes \mathrm{C}(\mathbb{T})$, $\mathcal{H} \subseteq C \otimes \mathrm{C}(\mathbb{T})$ be finite subsets, and let $\epsilon > 0$. Then there are $\delta > 0$ and a finite subset $\mathcal{G} \subseteq C$ such that for any C-algebra A which is tracially approximately divisible, any homomorphism $\varphi : C \to A$, any unitary $u \in A$ with*

$$\|[\varphi(c), u]\| < \delta \quad \forall c \in \mathcal{G},$$

there exist unitaries $w_1, w_2 \in A$, a path of unitaries $\{w(t); t \in [0,1]\} \subset A$ with $w(0) = 1$ and $w(1) = w_1w_2w_1^*w_2^* =: w$, and a completely positive $\mathcal{G}'$-ϵ-multiplicative linear maps $L_1, L_2 : C \otimes \mathrm{C}(\mathbb{T}) \to A$ such that

$$\|[w_i, \varphi(a)]\| < \epsilon \text{ for all } a \in \mathcal{F}, \ i = 1,2, \tag{e 3.5}$$
$$\|[w(t), \varphi(a)]\| < \epsilon, \quad \text{for all } a \in \mathcal{F}\cup\{u\} \text{ and } t \in [0,1], \tag{e 3.6}$$
$$\|L_1(a\otimes z) - (\varphi(a)uw)\| < \epsilon, \quad \|L_1(a\otimes 1) - \varphi(a)\| < \epsilon, \quad \text{for all } a \in \mathcal{F}, \tag{e 3.7}$$
$$\|L_2(a\otimes z) - (\varphi(a)w)\| < \epsilon, \quad \|L_2(a\otimes 1) - \varphi(a)\| < \epsilon, \quad \text{for all } a \in \mathcal{F}, \tag{e 3.8}$$
$$|\tau\circ L_1(g) - \tau\circ L_2(g)| < \epsilon, \quad \text{for all } g \in \mathcal{H}, \ \text{for all } \tau \in \mathrm{T}(A), \tag{e 3.9}$$

and

$$\mu_{\tau\circ L_i}(B_a) > \Delta_0(a), \quad i = 1,2, \text{ for all } \tau \in T(A)$$

and for any open ball B_a of $X\times\mathbb{T}$ with radius $a > 3\sqrt{2}\eta$.

Proof. Let $\tilde{\mathcal{H}} \subseteq C(X \times \mathbb{T})$ (in the place of $\mathcal{G}$) and $\tilde{\epsilon} > 0$ (in the place of δ) be the finite subset and constant of Lemma 3.4 of [14] with respect to $\Delta_{00}(a\sqrt{2}/2)a\sqrt{2}/8$, η and $\lambda_1 = \lambda_2 = 1/2$. Regarding $C(X \times \mathbb{T})$ as the center of $C \otimes C(\mathbb{T})$, the subset $\tilde{\mathcal{H}}$ is inside $C \otimes C(\mathbb{T})$.

Then without loss of generality, one may assume that $\tilde{\mathcal{H}} \subseteq \mathcal{H}$ and $\epsilon < \tilde{\epsilon}$, and one may also assume

$$\mathcal{G}' = \{f'_i \otimes z^{m_i};\ f'_i \in C, m_i \in \mathbb{Z}, i = 1, ..., N\},$$

$$\mathcal{H} = \{f_i \otimes z^{n_i};\ f_i \in C, n_i \in \mathbb{Z}, i = 1, ..., N\},$$

$1 \in \mathcal{F}$ and $\|f_i\|, \|f'_i\| \leq 1$. Choose $M \in \mathbb{N}$ so that $|m_i|, |n_i| < M$ for any $i = 1, ..., N$, and denote by

$$\mathcal{F}_1 = \{f'_i, f_i;\ i = 1, ..., N\}.$$

Let the natural number N_1 satisfies

$$\eta \in [\frac{1}{2^{N_1+1}}, \frac{1}{2^{N_1}}).$$

For each $1 \leq j \leq N_1$, by a compactness argument, choosing $\mathcal{O}_j$ to be a finite collection of open balls of X with radius $1/2^{j+2}$ which has the following property: for any open ball O_a of X with radius $a \in [1/2^{j+1}, 1/2^j)$, there is an open ball $O' \in \mathcal{O}_j$ such that $O' \subset O_a$.

Put $\mathcal{O} = \cup_{j=1}^{N_1} \mathcal{O}_j$. For each $O' \in \mathcal{O}_j$, fix a norm-one positive function g such that the support of $g_{O'}$ is in O', and is constant one if restricted to the open ball with the same center of O' and with the radius $\frac{1}{2^{j+3}}$. Then $g_{O'}P$ is a central element of C. Put $\mathcal{T} = \{g_{O'}P : O' \in \mathcal{O}\}$.

By Lemma 3.1, for any $\min\{\Delta(\frac{1}{2^{N_1+3}})/2^{N_1+7}, \epsilon/2\} > \epsilon' > 0$, there are $\delta' > 0$ and a finite subset $\mathcal{G} \subseteq C$ such that for any C*-algebra A, any unitary $v \in A$ with

$$\|[\varphi(c), v]\| < \delta', \quad \forall c \in \mathcal{G},$$

there exists a unital contractive completely positive linear map $L : C \otimes \mathrm{C}(\mathbb{T}) \to A$ with

$$\|L(f \otimes z^n) - \varphi(f)v^n\| < \epsilon' < \epsilon/16, \quad \forall f \in \mathcal{F} \cup \mathcal{F}_1,\ -M \leq n \leq M.$$

By choosing ϵ' sufficiently small, the resulting map L is $\mathcal{G}'$-ϵ-multiplicative. Without loss of generality, one may assume that $\delta' < \epsilon$.

One then asserts that $\delta := \delta'/2$ and $\mathcal{G}$ satisfy the lemma. Let $\varphi : C \to A$ be a homomorphism and $u \in A$ be a unitary with

$$\|[\varphi(c), u]\| < \delta, \quad \forall c \in \mathcal{G}.$$

Choose an integer $K \geq \max\{2^6\pi/\eta, 4(M+1)\}$. Since A is tracially approximately divisible, for any $\min\{\Delta(\frac{1}{2^{N_1+3}})/2^{N_1+7}, \epsilon/32M\} > \epsilon'' > 0$ (which will be fixed later), there is a projection $p \in A$, a unital C*-subalgebra $B \subset A$ with $B \cong M_k(\mathbb{C})$, with $1_B = p$ and $k \geq K$ such that

(1) $\tau(1-p) < \epsilon''/16$ for any $\tau \in T(A)$,

(2) $\varphi(\mathcal{F} \cup \mathcal{F}_1 \cup \mathcal{G} \cup \mathcal{T}) \subseteq_{\epsilon''} B' \cap A$ and $u \in_{\epsilon''} B' \cap A$,

where $B' \cap A$ is the relative commutant of B in A. Let $w' \in B \cong M_k(\mathbb{C})$ which has the following matrix form

$$w' = \begin{pmatrix} e^{2\pi i/k} & 0 & 0 & \cdots \\ 0 & e^{2\pi i 2/k} & 0 & \cdots \\ & & \ddots & \\ 0 & 0 & \cdots & e^{2\pi i k/k} \end{pmatrix}. \quad \text{(e 3.10)}$$

We compute that

$$t(w') = 0, \tag{e 3.11}$$

where $t \in T(B)$ is the tracial state. Moreover, for any $0 < |n| \leq M$,

$$t((w')^n) = 1 + \sum_{j=1}^{k-1} e^{2\pi nji/k} = \frac{1 - e^{2\pi nki/k}}{1 - e^{2\pi ni/k}} = 0. \tag{e 3.12}$$

In particular, $w' \in DU(B)$. Note that, since $B \cong M_k$, there exist two unitaries $w_1', w_2' \in B$ such that $w' = w_1' w_2' (w_1')^* (w_2')^*$. Let $\{w'(t); t \in [0,1]\} \subseteq B$ be a continuous path of unitaries such that $w'(0) = 1_B = p$ and $w'(1) = w'$. Denote by $w_1 = (1-p) + w_1'$, $w_2 = (1-p) + w_2'$ and

$$w = (1-p) + w' \quad \text{and} \quad w(t) = (1-p) + w(t).$$

It is clear that (e 3.5) holds when ϵ'' sufficiently small. By choosing ϵ'' smaller, it follows from (2) above that

$$\|[w(t), \varphi(a)]\| < \delta/2 < \epsilon, \quad \forall a \in \mathcal{F}, \forall t \in [0,1],$$

which also verifies (e 3.6).

One also assume that ϵ'' is even sufficiently small so that for any $c \in \mathcal{G}$

$$\|[\varphi(c), uw]\| < \delta', \quad \|[\varphi(c), w]\| < \delta', \text{ and } \|(uw)^n - u^n w^n\| < \epsilon/16, \quad -M \leq n \leq M. \tag{e 3.13}$$

Then there are $\mathcal{G}'$-ϵ-multiplicative linear maps $L_1, L_2 : C \otimes \mathrm{C}(\mathbb{T}) \to A$ such that

$$\|L_1(f \otimes z^n) - \varphi(f)(uw)^n\| < \epsilon' < \epsilon/16, \quad \forall f \in \mathcal{F} \cup \mathcal{F}_1, \ -M \leq n \leq M,$$

and

$$\|L_2(f \otimes z^n) - \varphi(f)w^n\| < \epsilon' < \epsilon/16, \quad \forall f \in \mathcal{F} \cup \mathcal{F}_1, \ -M \leq n \leq M.$$

Since $1 \in \mathcal{F}$, the maps L_1 and L_2 satisfy (e 3.7) and (e 3.8). Let us verify (e 3.9). Let τ be any tracial state of A. Note that, for any $a \in B \subseteq pAp$ and any $b \in B' \cap pAp$, one has that $\tau(ba) = \tau(b)\tau(a) = \tau(b)\tau(p)t(a)$, where t is the unique tracial state on B.

For any $i = 1, ..., N$, choose $a_i', u'' \in (1-p)A(1-p)$ and $a_i, u' \in B' \cap pAp$, where u', u'' are unitaries and $\|a_i\|$, $\|a_i'\| \leq 1$ such that

$$\|(a_i + a_i') - \varphi(f_i)\| < \epsilon'' < \epsilon/32 \quad \text{and} \quad \|(u' + u'') - u\| < \epsilon'' < \epsilon/32M.$$

Then

$$\begin{aligned}
\tau \circ L_1(f_i \otimes z^{n_i}) &\approx_{\epsilon/16} \tau(\varphi(f_i)(uw)^{n_i}) \\
&\approx_{\epsilon/16} \tau(\varphi(f_i)u^{n_i}w^{n_i}) \\
&\approx_{\epsilon/16} \tau(a_i(u')^{n_i}w^{n_i}) \\
&= \tau(a_i(u')^{n_i})t(w^{n_i}) \\
&\approx_{\epsilon/16} \begin{cases} \tau(\varphi(f_i)) & \text{if } n_i = 0 \\ 0 & \text{if } n_i \neq 0 \end{cases},
\end{aligned}$$

and

$$\begin{aligned}
\tau \circ L_2(f_i \otimes z^{n_i}) &\approx_{\epsilon/16} \tau(\varphi(f_i)w^{n_i}) \\
&\approx_{\epsilon/16} \tau(f_i' w^{n_i}) \\
&= \tau(a_i)t(w^{n_i}) \\
&\approx_{\epsilon/16} \begin{cases} \tau(\varphi(f_i)) & \text{if } n_i = 0 \\ 0 & \text{if } n_i \neq 0 \end{cases}.
\end{aligned}$$

Thus,

$$|\tau\circ L_1(f_i\otimes z^{n_i})-\tau\circ L_2(f_i\otimes z^{n_i})|<\epsilon.$$

Note that we choose $K\geq \max\{2^6\pi/\eta,\, 4(M+1)\}$. In particular,

$$\frac{2\pi}{K}\leq 1/2^{N_1+5}.$$

One then computes that

$$||\mu_t(I)-|I||<\frac{1}{2^{N_1+3}}$$

for any arc I with length at least η, where μ_t is the Borel probability measure induced by positive linear functional $t\circ f(w)$ on $C(\mathbb{T})$, where t is the tracial state on B.

Now, let B_a be any open ball on $X\times\mathbb{T}$ with radius a. Denote by (a_0,b_0) the center of B_a. Denote by $O_{a\sqrt{2}/2}$ the open ball of X with radius $a\sqrt{2}/2$ and center a_0, and denote by $J_{a\sqrt{2}/2}$ the open ball of $\mathbb{T}$ with radius $a\sqrt{2}/2$ and center b_0. Note that $O_{a\sqrt{2}/2}\times J_{a\sqrt{2}/2}\subseteq O_a$.

Assume that $a\sqrt{2}/2\in[\frac{1}{2^{j+1}},\frac{1}{2^j})$ for some $1\leq j\leq N_1$ Then choose $O_j'\in\mathcal{O}_j$ such that $O_j'\subseteq O'_{a\sqrt{2}/2}$, and consider $g_1P\in\mathcal{T}$ associated to O_k', and any norm-one positive continuous function g_2 on $\mathbb{T}$ with support in $J_{a\sqrt{2}/2}$. Note that

$$\varphi(g_1)\approx_{\epsilon''}a+b$$

for some b commutes with B and the traces of a are at most ϵ''.

Consider the function $g(x,t):=g_1(x)P\cdot g_2(t)$. Then, for any $a\geq\sqrt{2}\eta>\eta$,

$$\begin{aligned}
\mu_{\tau\circ L_2}(B_a)\geq\tau(L_2(g)) &> \tau(bg_2(w))-\epsilon' && \text{(e 3.14)}\\
&= \tau(b)\cdot t(g_2(w))-\epsilon' && \text{(e 3.15)}\\
&> \tau(\varphi(g_1))\cdot t(g_2(w))-\epsilon'-\epsilon'' && \text{(e 3.16)}\\
&> \Delta(\frac{1}{2^{k+3}})\cdot t(g_2(w))-\epsilon'-\epsilon'' && \text{(e 3.17)}
\end{aligned}$$

Since this holds for any g_2, one has

$$\mu_{\tau\circ L_2}(B_a)\geq\Delta(\frac{1}{2^{k+3}})\cdot\mu_t(J_{a\sqrt{2}/2})/2-\Delta(\frac{1}{2^{N_1+3}})/2^{N_1+5}=\Delta(\frac{1}{2^{k+3}})a\sqrt{2}/8>\Delta_{00}(a\sqrt{2}/2)a\sqrt{2}/8,$$

where μ_t is the spectral measure of w.

Note that

$$|\tau\circ L_1(g)-\tau\circ L_2(g)|<\epsilon<\tilde{\epsilon}\quad\forall g\in\tilde{\mathcal{H}}\subseteq\mathcal{H},$$

by Lemma 3.4 of [14], one has

$$\mu_{\tau\circ L_1}(B_a)>\frac{\sqrt{2}}{48}\Delta_{00}(a\sqrt{2}/6)a=\Delta_0(a)$$

for any $a\geq 3\sqrt{2}\eta$. □

Definition 3.8. Let A be a unital C*-algebra. In the following, for any invertible element $x\in A$, let $\langle x\rangle$ denote the unitary $x(x^*x)^{-\frac{1}{2}}$, and let $\overline{x}$ denote the element $\overline{\langle x\rangle}$ in $U(A)/CU(A)$. Consider a subgroup $\mathbb{Z}^k\subseteq K_1(A)$, and write the unitaries $\{u_1,...,u_k\}\subseteq U_c(A)$ corresponding to the standard generators $\{e_1,e_2,...,e_k\}$ of $\mathbb{Z}^k$. Suppose that $\{u_1,u_2,...,u_k\}\subset M_n(A)$ for some integer $n\geq 1$. Let $\Phi: A\to B$ be a unital positive linear map such that $\Phi\otimes\mathrm{id}_{M_n}$ is at least $\{u_1, ..., u_k\}$-1/4-multiplicative (hence each $\Phi \otimes \mathrm{id}_{M_n}(u_i)$ is invertible), then the map $\Phi^{\ddagger}|_{s_1(\mathbb{Z}^k)} : \mathbb{Z}^k \to U(B)/CU(B)$ is defined by

$$\Phi^{\ddagger}|_{s_1(\mathbb{Z}^k)}(e_i) = \overline{\langle \Phi \otimes \mathrm{id}_{M_n}(u_i) \rangle}, \quad 1 \le i \le k.$$

Thus, for any finitely generated subgroup $G \subset \overline{U_c(A)}$, there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ such that, for any unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $L : A \to B$ (for any unital C*-algebra B), the map $L^{\ddagger}$ is well defined on $s_1(G)$. (Please see 2.4 for $U_c(A)$ and s_1.)

Theorem 3.9. *Let $C = \mathrm{C}(X)$ with X a compact metric space and let $\Delta : (0,1) \to (0,1)$ be a non-decreasing map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subseteq C$, there exists $\delta > 0$, $\eta > 0$, $\gamma > 0$, a finite subsets $\mathcal{G} \subseteq C$, $\mathcal{P} \subseteq \underline{K}(C)$, a finite subset $\mathcal{Q} = \{x_1, x_2, ..., x_m\} \subset K_0(C)$ which generates a free subgroup and $x_i = [p_i] - [q_i]$, where $p_i, q_i \in M_n(C)$ (for some integer $n \ge 1$) are projections, satisfying the following:*

Suppose that A is a unital simple C-algebra with $TR(A) \le 1$, $\varphi : C \to A$ is a unital homomorphism and $u \in A$ is a unitary, and suppose that*

$$\|[\varphi(c), u]\| < \delta, \ \forall c \in \mathcal{G} \quad \text{and} \quad \mathrm{Bott}(\varphi, u)|_{\mathcal{P}} = 0,$$

and

$$\mu_{\tau\circ\varphi}(O_a) \ge \Delta(a) \ \forall \tau \in T(A),$$

where O_a is any open ball in X with radius $\eta \le a < 1$ and $\mu_{\tau\circ\varphi}$ is the Borel probability measure defined by $\tau \circ \varphi$. Moreover, for each $1 \le i \le m$, there is $v_i \in CU(M_n(A))$ such that

$$\|\langle(\mathbf{1}_n - \varphi(p_i) + \varphi(p_i)1_n \otimes u)(\mathbf{1}_n - \varphi(q_i) + \varphi(q_i)(1_n \otimes u^*\rangle - v_i\| < \gamma.$$

Then there is a continuous path of unitaries $\{u(t) : t \in [0,1]\}$ in A such that

$$u(0) = u, u(1) = 1, \ \text{and} \ \|[\varphi(c), u(t)]\| < \epsilon$$

for any $c \in \mathcal{F}$ and for any $t \in [0,1]$.

Proof. Since A is a simple C*-algebra with $TR(A) \le 1$, it is tracially approximately divisible (see 5.4 of [8]). Therefore 3.7 applies. Without loss of generality, one may assume that $\mathcal{F}$ is in the unit ball of C. Let ϵ_0 be the universal constant such that, for any unitaries u_1 and u_2 with $\|u_1 - u_2\| < \epsilon_0$, there is a path of unitaries connecting u_1 and u_2 with length at most $\epsilon/2$.

Let $\eta' > 0$, $\delta' > 0$, $\mathcal{G}' \subseteq C \otimes \mathrm{C}(\mathbb{T})$, $\mathcal{H} \subseteq C \otimes \mathrm{C}(\mathbb{T})$, $\mathcal{P}' \subseteq \underline{K}(C \otimes \mathrm{C}(\mathbb{T}))$, $\mathcal{U}' \subseteq U_c(K_1(C \otimes \mathrm{C}(\mathbb{T})))$, γ_1, and γ_2 be the finite subsets and constants of Theorem 5.3 of [14] with respect to $X \times \mathbb{T}$, Δ_0, $\mathcal{F} \otimes \{1, z\}$, and $\min\{\epsilon/2, \epsilon_0\}$. Without loss of generality, we may assume that $\mathcal{P}' = \mathcal{P} \cup \boldsymbol{\beta}(\mathcal{P})$, where $\mathcal{P}$ is a finite subset of $\underline{K}(C)$, and

$$\mathcal{G}' = \mathcal{G}_1' \cup \{1_{\mathrm{C}(\mathbb{T})}, z\},$$

where $\mathcal{G}_1'$ is a finite subset of C. Moreover, we may assume

$$[L']|_{\mathcal{P}} = [L'']|_{\mathcal{P}} \tag{e 3.18}$$

for any unital $\mathcal{G}_1'$-δ'-multiplicative contractive completely positive linear maps $L', L'' : C \to A$ with

$$\|L'(g) - L''(g)\| < \gamma_2 \ \text{ for all } \ g \in \mathcal{G}_1'.$$

By choosing larger $\mathcal{G}_1'$ and smaller δ', we may assume further that $(L')^{\ddagger}$ is well defined on $\overline{\mathcal{U}'}$.

Since $K_1(C \otimes \mathrm{C}(\mathbb{T})) = K_1(C) \oplus K_0(C)$, without of loss of generality, the set $\mathcal{U}'$ may be chosen as $\mathcal{U}'_1 \cup \mathcal{U}'_0$, where $\mathcal{U}'_1 = \{\overline{v_1 \otimes 1_{\mathrm{C}(\mathbb{T})}}, ..., \overline{v_{n'} \otimes 1_{\mathrm{C}(\mathbb{T})}}\}$ with each v_i a unitary $M_n(C)$, and any element in $\mathcal{U}'_0$ has the form

$$\overline{(p \otimes z + (\mathbf{1}_n - p) \otimes 1_{\mathrm{C}(\mathbb{T})})(q \otimes z + (\mathbf{1}_n - q) \otimes 1_{\mathrm{C}(\mathbb{T})})^*}$$

for some projections p and q in $M_n(C)$ for some integer $n \geq 1$. Without loss of generality, one may assume that $\mathcal{U}'_0$ exactly generates a free group $\mathbb{Z}^m$ in $K_1(C \otimes \mathrm{C}(T))$ as standard generators, and hence one may write

$$\mathcal{U}'_0 = \{\overline{(p_i \otimes z + (\mathbf{1}_n - p_i) \otimes 1_{\mathrm{C}(\mathbb{T})})(q_i \otimes z + (\mathbf{1}_n - q_i) \otimes 1_{\mathrm{C}(\mathbb{T})})^*};\ i = 1, ..., m\},$$

where p_i and q_i are projections in $M_n(C)$. Denote by $x_i = [p_i] - [q_i]$ for $1 \leq i \leq m$, and put $\mathcal{Q} = \{x_1, ..., x_m\}$.

We may assume that $\mathcal{F}_1 \subset C$ is a finite subset such that

$$p_i, q_i \in \{(c_{j,k}) \in M_n(C) : c_{j,k} \in \mathcal{F}_1\}.$$

Put $\mathcal{F}_2 = \{1_C\} \cup \mathcal{F} \cup \mathcal{F}_1$. Let $\delta > 0$ and $\mathcal{G} \subseteq C$ be the constant and finite subset of Lemma 3.7 with respect to $\min\{\epsilon/8n^2, \delta'/n^2, \gamma_1/2n^2, \gamma_2/16n^2\}$ (in place of ϵ), $\mathcal{F}_2$ (in place of $\mathcal{F}$), $\mathcal{G}'$ and $\mathcal{H}$.

Without loss of generality, one may assume that δ is sufficiently small and $\mathcal{G}$ is sufficiently large such that $\mathrm{Bott}(\varphi, u_1u_2)|_{\mathcal{P}}$ is well defined and

$$\mathrm{Bott}(\varphi, u_1u_2)|_{\mathcal{P}} = \mathrm{Bott}(\varphi, u_1)|_{\mathcal{P}} + \mathrm{Bott}(\varphi, u_2)|_{\mathcal{P}}$$

for any unital homomorphisms $\varphi : C \to B$ for some unital C*-algebra B and unitaries $u_1, u_2 \in B$ with

$$\|[\varphi(a), u_i]\| < \delta, \quad \forall a \in \mathcal{G}, i = 1, 2.$$

One asserts that δ, $\eta = \frac{\sqrt{2}}{6}\eta'$, $\gamma = \gamma_2/4$, $\mathcal{P}$, $\mathcal{G}$ and $\mathcal{Q}$ satisfy the theorem.

Let (φ, u) be a pair which satisfies the condition of the theorem. By Lemma 3.7, there are unitary $w = w_1w_2w_1^*w_2^*$ with w_1, w_2 unitaries in A, a path of unitaries $\{w'(t); t \in [0,1]\}$ with $w'(1) = 1$ and $w'(0) = w$, and unital $\mathcal{G}'$-δ'-multiplicative completely positive linear maps $L_1, L_2 : C \otimes \mathrm{C}(\mathbb{T}) \to A$ such that for any $f \in \mathcal{F}$,

(1) $\|[w_i, \varphi(a)]\| < \min\{\epsilon/8n^2, \gamma_2/16n^2\}$, $\forall a \in \mathcal{F}_2$, i=1, 2,

(2) $\|[w'(t), \varphi(a)]\| < \min\{\epsilon/8n^2, \gamma_2/8n^2\}$, $\forall a \in \mathcal{F}_2 \cup \{u\}$, $\forall t \in [0,1]$,

(3) $\|L_1(f \otimes z) - (\varphi(f)uw)\| < \min\{\epsilon/8n^2, \gamma_2/8n^2\}$, $\quad \|L_1(f \otimes 1) - \varphi(f)\| < \min\{\epsilon/8n^2, \gamma_2/8n^2\}$,

(4) $\|L_2(f \otimes z) - (\varphi(f)w)\| < \min\{\epsilon/8n^2, \gamma_2/8n^2\}$, $\quad \|L_2(f \otimes 1) - \varphi(f)\| < \min\{\epsilon/8n^2, \gamma_2/8n^2\}$,

(5) $|\tau \circ L_1(g) - \tau \circ L_2(g)| < \gamma_1/2n^2$, $\quad \forall g \in \mathcal{H}$, $\forall \tau \in \mathrm{T}(A)$,

(6) $\mu_{\tau \circ L_i}(O_a) > \Delta_0(a)$, $i = 0, 1$ for any open ball O_a of $X \times \mathbb{T}$ with radius $a > 3\sqrt{2}\eta = \eta'$,

It follows from (2) that $\mathrm{Bott}(\varphi, w) = 0$. Therefore

$$\begin{aligned} \mathrm{Bott}(\varphi, uw)|_{\mathcal{P}} &= \mathrm{Bott}(\varphi, u)|_{\mathcal{P}} + \mathrm{Bott}(\varphi, w)|_{\mathcal{P}} \\ &= \mathrm{Bott}(\varphi, u)|_{\mathcal{P}} = 0. \end{aligned}$$

We also have, by (e 3.18),

$$[L_1]|_{\mathcal{P}} = [\varphi]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}}. \tag{e 3.19}$$

Note that, by (1),

$$w = w_1 w_2 w_1^* w_2^*$$

with $\|[w_i, \varphi(a)]\| < \min\{\epsilon/8n^2, \gamma_2/16n^2\}$, $\forall a \in \mathcal{F}_2$, $i = 1, 2$. Then for any projection p_i (or q_i), one estimates that

$$\operatorname{dist}(\langle(\mathbf{1}_n - \varphi(p_i)) + \varphi(p_i)w\rangle, CU(M_n(A))) < \gamma_2/16 \text{ and} \tag{e 3.20}$$
$$\operatorname{dist}(\langle(\mathbf{1}_n - \varphi(q_i)) + \varphi(q_i)w\rangle, CU(M_n(A))) < \gamma_2/16, \tag{e 3.21}$$

$1 \le i \le m$. Therefore, for any $1 \le i \le m$,

$$\operatorname{dist}(\overline{L_2((p_i \otimes z + \mathbf{1}_n - p_i)(q_i \otimes z + \mathbf{1}_n - q_i)^*)}, \overline{1}) \approx_{\gamma_2/4} 0, \text{ and} \tag{e 3.22}$$
$$\operatorname{dist}(\overline{\langle L_1((p_i \otimes z + (\mathbf{1}_n - p_i))(q_i \otimes z + (\mathbf{1}_n - q_i))^*)\rangle}, \overline{\mathbf{1}_n}) \tag{e 3.23}$$
$$\approx_{\gamma_2/8} \operatorname{dist}(\overline{\langle((\mathbf{1}_n - \varphi(p_i)) + \varphi(p_i)uw)((\mathbf{1}_n - q_i) + \varphi(q_i)uw)^*\rangle}, \overline{\mathbf{1}_n}) \tag{e 3.24}$$
$$\approx_{\gamma_2/8} \operatorname{dist}(\overline{\langle((\mathbf{1}_n - \varphi(p_i)) + \varphi(p_i)u)((\mathbf{1}_n - q_i) + \varphi(q_i)u)^*\rangle}, \overline{\mathbf{1}_n}) \tag{e 3.25}$$
$$< \gamma = \gamma_2/4. \tag{e 3.26}$$

Also note that for any $v_i \otimes 1 \in \mathcal{U}_1'$, one computes that

$$\operatorname{dist}(\overline{\langle L_1(v_i \otimes 1)\rangle}, \overline{L_2(v_i \otimes 1)}) \approx_{\gamma_2} \operatorname{dist}(\overline{\varphi(v_i)}, \overline{\varphi(v_i)}) = 0.$$

Since $U_0(A)/CU(A)$ is torsion free (Theorem 6.11 of [8]), one has that

$$\operatorname{dist}(\overline{\langle L_1(u)\rangle}, \overline{\langle L_2(u)\rangle}) < \gamma_2, \quad \forall u \in \mathcal{U}'. \tag{e 3.27}$$

By (e 3.19) (5), (e 3.27) and (6), it follows from Theorem 5.3 of [14] that there is a unitary $U \in A$ such that

$$\|L_1(f) - U^* L_2(f) U\| < \min\{\epsilon/2, \epsilon_0\}, \quad \forall f \in \mathcal{F} \otimes \{1, z\}.$$

Consider the path of unitaries $w(t) : t \mapsto U^* w'(2t-1)U$, $t \in [1/2, 1]$. Then

$$\|[\varphi(f), w(t)]\| < \epsilon \quad \forall f \in \mathcal{F}, t \in [1/2, 1] \text{ and } \|w(1/2) - uw\| < \epsilon_0, \quad w(1) = 1. \tag{e 3.28}$$

By the choice of ϵ_0, there is a path of unitaries $\{w''(t);\ t \in [1/4, 1/2]\}$ such that

$$\|[\varphi(f), w(t)]\| < \epsilon, \quad \forall f \in \mathcal{F},\ t \in [1/4, 1/2], \text{ and} \tag{e 3.29}$$
$$w''(1/4) = uw \quad \text{and} \quad w''(1/2) = w(1/2). \tag{e 3.30}$$

Also consider the path of unitaries $w'''(t) : t \to uw'(4t)$, $t \in [0, 1/4]$. Then one has that $w'''(0) = u$, $w'''(1/4) = uw$ and

$$\|[w'''(t), \varphi(f)]\| < \epsilon, \quad \forall f \in C.$$

Define the path

$$w(t) = \begin{cases} w'''(t), & \text{if } t \in [0, 1/4], \\ w''(t), & \text{if } t \in [1/4, 1/2], \\ w(t), & \text{if } t \in [1/2, 1]. \end{cases}$$

Then it is clear that

$$\|[\varphi(f), w(t)]\| < \epsilon, \quad \forall f \in \mathcal{F}, t \in [0, 1],$$

$$w(0) = u \quad \text{and} \quad w(1) = 1,$$

as desired. $\square$

Corollary 3.10. *Let X be a compact subset of finite CW-complex and let $C = PM_n(\mathrm{C}(X))P$ for some integer $n \geq 1$ and P a projection in $M_n(\mathrm{C}(X))$. Then the statement of Theorem 3.9 still holds for the C*-algebra C and using the measure define in 3.4.*

Proof. If $C = M_n(C(X))$, it is clear that the corollary follows from Theorem 3.9 (X is even not required to have finite covering space in this case). In what follows we will use this case of the corollary to prove the general case.

Assume that $C = PM_n(C(X))P$. Since X is compact, the rank of P has only finitely many values. It follows that, without loss of generality, we may assume that $P(x) \neq 0$ for all $x \in X$. Since X is a compact subset of finite CW-complex, there is an integer d and a projection $Q \in M_d(PM_n(C(X))P)$ such that

$$QM_d(PM_n(C(X))P)Q \cong M_r(C(X))$$

for some integer r. Note that $Q(x) \neq 0$ for all $x \in X$. Without loss of generality, one may assume that $P \preceq Q$, that is, there is also a partial isometry $V \in M_d(PM_n(C(X))P)$ such that $VV^* \leq Q$ and $V^*V = \{P, 0, ..., 0\}$.

There is an integer $l \geq 1$ such that $X = X_1 \sqcup \cdots \sqcup X_l$ such that the ranks of the restrictions of P and Q to each X_i, $1 \leq j \leq l$, are constant. Denote by P_j and Q_j the restriction of P and Q to X_j respectively. Let $R_1 = \max_{1\leq j\leq l}\{\mathrm{rank}P_j\}$ and $R_2 = \min_{1\leq j\leq l}\{\mathrm{rank}Q_j\}$.

Fix d, Q, and V. Let $\Delta : (0, 1) \to (0, 1)$ be a non-decreasing map, let $\epsilon > 0$ and $\mathcal{F} \subseteq PM_n(C(X))P$ be a finite subset of elements with norm one.

Pick $\frac{\epsilon}{4} > \epsilon' > 0$ such that for any unitaries u, v in a C*-algebra with $\|u - v\| < \epsilon'$, there is a path of unitaries $u(t)$ such that $u(0) = u$, $u(1) = v$, and $\|u(t) - v\| < \frac{\epsilon}{2}$, $\forall t \in [0, 1]$.

Pick $\frac{\epsilon'}{4} > \epsilon'' > 0$ such that if there are a projection p and a unitary U in a C*-algebra A with $\|[p, U]\| < \epsilon''$, then

$$\|\langle pUp\rangle - pUp\| < \epsilon'/4.$$

(Recall that $\langle pUp\rangle = pUp(pU^*pUp)^{-\frac{1}{2}}$.)

Denote by δ' (in place δ), η, γ' (in place of γ), $\mathcal{G}' \subseteq QM_d(P(M_n(C(X)))P)Q \cong M_r(C(X))$ (in place of $\mathcal{G}$), $\mathcal{P} \subseteq \underline{K}(C(X))$, and $\mathcal{Q} \subseteq K_0(C(X))$ the constants and finite subsets of the corollary required for $M_r(C(X))$ with ϵ'', $V\mathcal{F}V^*$, and Δ.

We may assume that $\gamma' < 1$. For each $x_i \in \mathcal{Q}$, write $x_i = [p_i]-[q_i]$ with $p_i, q_i \in M_k(QM_d(C)Q)$ for some integer k. Choose an integer k' such that

$$M_k(QM_d(C)Q) \subseteq M_{k'}(C).$$

Without loss of generality, one also assumes that any element of $\mathcal{G}'$ has norm one, and $VV^* \in \mathcal{G}'$. Choose a finite subset $\mathcal{G}_1 \subseteq C$ and $\delta_1 > 0$ such that if there is a C*-algebra A and a unitary $u \in A$ satisfies

$$\|[\varphi(c), u]\| < \delta_1, \quad \forall c \in \mathcal{G}_1$$

for some homomorphism φ to A, then

$$\|[(\varphi \otimes \mathrm{id}_{M_d})(c), u \otimes 1_{M_d}]\| < \delta'/2$$

for any $c \in \mathcal{G}' \subseteq M_d(C)$, and

$$\|[\varphi \otimes \mathrm{id}_{M_d}(Q), u \otimes 1_{M_d}]\| < \min\{\epsilon'', \delta'/2\}.$$

Let $B = QM_d(C)Q \otimes C(\mathbb{T})$. It is a full hereditary C*-subalgebra of $M_d(C) \otimes C(\mathbb{T})$. Choose a large finite subset $\mathcal{G}_2 \subset C$ and a sufficiently small $\delta_2 > 0$ such that, if $L : M_d(C)\otimes C(\mathbb{T}) \to M_d(A)$

is a unital $\mathcal{G}_2 \times \{1, z\}$-$\delta_2$-multiplicative contractive completely positive linear map and $[L]|_{\mathcal{P}}$ is well defined, then

$$[L]|_{\mathcal{P}} = [L|_B]|_{\mathcal{P}}.$$

Note that if we assume that

$$\mathrm{Bott}(\varphi, u)|_{\mathcal{P}} = 0, \tag{e 3.31}$$

then

$$\mathrm{Bott}(\varphi \otimes \mathrm{id}_{M_d}, u \otimes 1_{M_d})|_{\mathcal{P}} = 0. \tag{e 3.32}$$

It then follows that we can choose a larger $\mathcal{G}_2$ and smaller δ_2 so that if

$$\|[\varphi(c), u]\| < \delta_2, \quad \forall c \in \mathcal{G}_2 \ \text{ and } \ \mathrm{Bott}(\varphi, u)|_{\mathcal{P}} = 0, \tag{e 3.33}$$

we still have

$$\mathrm{Bott}(\varphi|_{QM_d(C)Q}, \langle q(u \otimes 1_{M_d})q\rangle)|_{\mathcal{P}} = 0, \tag{e 3.34}$$

where $q = (\varphi \otimes \mathrm{id}_{M_d})(Q)$.

Note that $p_i, q_i \in M_k(QM_d(C)Q) \subseteq M_{k'}(C)$. Define $\bar{q} = q \otimes 1_{M_k}$. Then there is a finite subset $\mathcal{G}_3 \subseteq C$, and $\delta_3 > 0$ such that if

$$\|[\varphi(c), u]\| < \delta_3, \quad \forall c \in \mathcal{G}_3 \ \text{ and} \tag{e 3.35}$$

$$\|\langle(1_{M_{k'}} - \varphi(p_i) + \varphi(p_i)1_{M_{k'}} \otimes u)(1_{M_{k'}} - \varphi(q_i) + \varphi(q_i)1_{M_{k'}} \otimes u^*))\rangle - v_i\| < \gamma'/(8(k'R_1 + \frac{1}{8})),$$

for some $v_i \in CU(M_{k'}(A))$, then

$$\|g_i - ((1_{M_{k'}} - \bar{q}) + \langle \bar{q} g_i \bar{q}\rangle)\| < \gamma'/(4(k'R_1 + \frac{1}{8})), \tag{e 3.36}$$

and

$$\|g_i' - \langle \bar{q} g_i \bar{q}\rangle\| < \gamma'/(4(k'R_1 + \frac{1}{8})), \tag{e 3.37}$$

where

$$g_i := \langle(1_{M_{k'}} - \varphi(p_i) + \varphi(p_i)u \otimes 1_{M_{k'}})(1_{M_{k'}} - \varphi(q_i) + \varphi(q_i)u^* \otimes 1_{M_{k'}})\rangle$$

and

$$g_i' := \langle(\bar{q} - \varphi(p_i) + \varphi(p_i)\langle \bar{q}(u \otimes 1_{M_{k'}})\bar{q}\rangle)(\bar{q} - \varphi(q_i) + \varphi(q_i)\langle \bar{q}(u^* \otimes 1_{M_{k'}})\bar{q}\rangle)\rangle.$$

Note that, in particular, one has

$$\|g_i - ((1_{M_{k'}} - \bar{q}) + g_i')\| < \gamma'/2(k'R_1 + \frac{1}{8}). \tag{e 3.38}$$

Then

$$\mathrm{dist}((1_{M_{k'}} - \bar{q}) + g_i', CU(M_{k'}(A))) < \gamma'/(k'R_1 + \frac{1}{8}). \tag{e 3.39}$$

Since $TR(A) \le 1$, it follows from Lemma 6.9 of [8] that $CU(M_{k'}(A)) \subset U_0(M_{k'}(A))$. It follows from the fact that $\gamma' < 1$ and (e 3.39) that $(1_{M_{k'}} - \bar{q}) + g_i' \in U_0(M_{k'}(A))$. Since A is a unital simple C*-algebra with $\mathrm{TR}(A) \le 1$, one has that $g_i' \in U_0(M_k(qM_d(A)q))$ (see 2.10 of [19]). Note that for any $\tau \in T(M_{k'}(A))$, one has

$$\tau(\bar{q}) \ge \frac{kR_2}{k'R_1} > \frac{1}{k'R_1},$$

and hence

$$k'R_1[\bar{q}] > [1_{M'_k} - \bar{q}].$$

Then by Lemma 3.3 of [11], one has

$$\mathrm{dist}(g'_i, CU(M_k(qM_d(A)q))) < (k'R_1 + \frac{1}{8})\frac{\gamma'}{(k'R_1 + \frac{1}{8})} = \gamma'.$$

That is,

$$\|\langle(\bar{q} - \varphi(p_i) + \varphi(p_i)\langle\bar{q}(u \otimes 1_{M_d})\bar{q}\rangle)(\bar{q} - \varphi(q_i) + \varphi(q_i)\langle\bar{q}(u^* \otimes 1_{M_d})\bar{q}\rangle)\rangle - v'_i\| < \gamma', \qquad (\mathrm{e}\,3.40)$$

for some $v'_i \in CU(M_k(qM_d(A)q))$.

Put $\gamma = \gamma'/(8(k'R_1 + \frac{1}{8}))$. Then, one asserts that $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, η, γ, $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$, $\mathcal{P}$, and $\mathcal{Q}$ satisfy the corollary.

Let $\varphi : PM_n(C(X))P \to A$ be a unital homomorphism satisfies the conditions of the corollary for some unitary $u \in A$, where A is a simple C*-algebra with $TR(A) \leq 1$.

Put $v = \varphi \otimes 1_{M_d}(V) \in M_d(A)$. The restriction of $\varphi \otimes 1_{M_d}$ to $QM_d(C)Q$ (which is isomorphic to $M_r(C(X))$) is a unital homomorphism to $qM_d(A)q$, which has $TR(qM_d(A)q) \leq 1$, and one also has that $vv^* \leq q$ and $v^*v = 1_A$.

Since $\|[\varphi(c), u]\| < \delta_1$, $\forall c \in \mathcal{G}_1$, one has

$$\|[(\varphi \otimes 1_{M_d})(c), u \otimes 1_{M_d}]\| < \delta'/2, \quad \forall c \in \mathcal{G}'.$$

In particular,

$$\|[(\varphi \otimes 1_{M_d})|_{QM_d(C)Q}(c), \langle q(u \otimes 1_{M_d})q\rangle]\| < \delta', \quad \forall c \in \mathcal{G}'.$$

Since φ also satisfies (e 3.33) (and (e 3.35)), Equations (e 3.34) (and (e 3.40)) are also satisfied.

Since $\mu_{\tau\circ\varphi}(O_a) \geq \Delta(a)$ for any open ball O_a on X with radius $1 > a > \eta$ and any $\tau \in T(A)$, one then also has that

$$\mu_{\tau\circ((\varphi\otimes 1_{M_d})|_{QM_d(C)Q})}(O_a) \geq \Delta(a)$$

for any open ball O_a on X with radius $1 > a > \eta$ and any tracial state τ on $qM_d(A)q$.

Then, applying the corollary to $QM_d(C)Q$ and $qM_d(A)q$, there is a path of unitaries $\{U(t);\ t \in [0,1]\} \subseteq qM_d(A)q$ such that

$$U(0) = 1_{qM_d(A)q}, \quad U(1) = \langle q(u \otimes 1_{M_d})q\rangle,$$

and

$$\|[\varphi \otimes 1_{M_d}(VfV^*), U(t)]\| < \epsilon'', \quad \forall f \in \mathcal{F}.$$

Denote by $e = vv^* \in qM_d(A)q$. Note that $\|[e, U(t)]\| < \epsilon'' < \frac{1}{4}$. One considers the path of unitaries

$$w(t) = \langle eU(t)e\rangle \in eM_d(A)e, \quad t \in [0,1].$$

Then

$$w(0) = r, \quad \|w(1) - e(u \otimes 1_{M_d})e\| < \epsilon'/2,$$

$$\|[v(\varphi \otimes 1_{M_d})(f)v^*, w(t)]\| < 2\epsilon' + 2\epsilon'' < \epsilon, \quad \forall f \in \mathcal{F}.$$

Consider the path of unitaries $u(t) := v^*w(t)v \in A$. One then has that

$$u(0) = 1_A, \quad \|u(1) - u\| < \epsilon'/2 + \epsilon'' < \epsilon',$$

and

$$\|[\varphi(f), u(t)]\| < \epsilon, \quad \forall f \in \mathcal{F}.$$

□

Remark 3.11. In fact, the corollary above holds for the case that X is a general compact metric space. One can use a standard argument reducing the general case to the case that X is a compact subset of a finite CW-complex.

The following lemma is due to N.C. Phillips. (See the proof of 3.8 of [18].)

Lemma 3.12. *Let A be a unital C^*-algebra and $2 > d > 0$. Let $u_0, u_1, ..., u_n$ be $n+1$ unitaries in A such that*

$$u_n = 1 \quad \text{and} \quad \|u_i - u_{i+1}\| \le d \quad i = 0, 1, ..., n-1.$$

Then there exists a unitary $v \in CU(\mathrm{M}_{2n+1}(A))$ with exponential length no more than 2π such that

$$\|(u_0 \oplus 1_{\mathrm{M}_{2n}(A)}) - v\| \le d.$$

In the rest of the paper, unless otherwise specified, z will be the identity function on the unit circle.

Theorem 3.13. *Let $C = \mathrm{C}(X)$ with X a compact metric space, let $G \subset K_0(C)$ be a finitely generated subgroup. Write $G = \mathbb{Z}^k \oplus \mathrm{Tor}(G)$ with $\mathbb{Z}^k$ generated by*

$$\{x_1 = [p_1] - [q_1], x_2 = [p_2] - [q_2], ..., x_k = [p_k] - [q_k]\},$$

where $p_i, q_i \in M_n(\mathrm{C}(X))$ (for some integer $n \ge 1$) are projections, $i = 1, ..., k$.

Let A be a simple C^*-algebra with $TR(A) \le 1$. Suppose that $\varphi : C \to A$ is a monomorphism. Then, for any finite subsets $\mathcal{F} \subseteq C$ and $\mathcal{P} \subseteq \underline{K}(C)$, any $\epsilon > 0$ and $\gamma > 0$, any homomorphism

$$\Gamma : \mathbb{Z}^k \to U_0(A)/CU(A),$$

there is a unitary $w \in A$ such that

$$\|[\varphi(f), w]\| < \epsilon \quad \forall f \in \mathcal{F}, \ \ \mathrm{Bott}(\varphi, w)|_{\mathcal{P}} = 0, \ \text{ and} \tag{e 3.41}$$

$$\mathrm{dist}(\overline{\langle(\mathbf{1}_n - \varphi(p_i) + \varphi(p_i)w)(\mathbf{1}_n - \varphi(q_i) + \varphi(q_i)w^*\rangle}, \Gamma(x_i)) < \gamma, \quad \forall 1 \le i \le k,$$

where $U_0(A)/CU(A)$ is identified as $U_0(M_m(A))/CU(M_m(A))$, and the distance above is understood as the distance in $U_0(M_m(A))/CU(M_m(A))$.

Proof. Without loss of generality, we may assume that $\epsilon < 1/2$. Denote by

$$\Delta(a) = \inf\{\mu_{\tau\circ\varphi}(O_a);\ \tau \in \mathrm{T}(A), O_a \text{ is an open ball of } X \text{ with radius } a\}.$$

Since A is simple and $\mathrm{T}(A)$ is compact, $\Delta(a)$ is a nondecreasing function from $(0,1)$ to $(0,1)$.

Let $\eta' > 0$, $\delta' > 0$, $\mathcal{G}' \subseteq C$, $\mathcal{H}' \subseteq C_{s.a.}$, $\mathcal{P}' \subseteq \underline{K}(C)$, $\mathcal{U}' \subset U_c(K_1(C))$, $\gamma_1 > 0$, $\gamma_2 > 0$ be the finite subsets and constants of Theorem 5.3 of [14] with respect to X, $\Delta(r/3)/2$, $\mathcal{F}$, and $\epsilon/2$. Without loss of generality, one may assume that $\mathcal{F} \subseteq \mathcal{G}'$ and $\delta' < \epsilon/4$.

Let δ'' and $\mathcal{H}'' \subseteq C$ be the constant and finite subset of lemma 3.4 of [14] with respect to X, Δ, and $\eta'/3$.

Since X is an inverse limit of finite CW-complexes, there is a C*-algebra $C' \cong \mathrm{C}(X')$ for a finite CW-complex X' and a homomorphism $\iota : C' \to C$ such that

$$G \subseteq \iota_{*0}(K_0(C')), \qquad \{p_i, q_i;\ i = 1, ..., k\} \subseteq \iota(M_n(C')), \ \text{ and} \tag{e 3.42}$$

$$\mathcal{P}' \subset [\imath](\mathcal{P}_1'), \tag{e 3.43}$$

where $\mathcal{P}_1' \subset \underline{K}(C')$ is a finite subset.

Furthermore, one may choose X' such that there is a completely positive linear map $\pi : C \to C'$ so that if $\psi : C' \to A$ is $(\mathcal{G}'')$-$\delta'/2$-multiplicative (for some finite subset $\mathcal{G}'' \subset C'$,) then $\psi \circ \pi$ is $\mathcal{G}'$-δ'-multiplicative, and moreover,

$$\|\imath \circ \pi(f) - f\| < \min\{\epsilon/8, \gamma_1/4\}, \quad \forall f \in \mathcal{F} \cup \mathcal{H}' \cup \mathcal{H}'',$$

and $[\pi](\mathcal{P}') \subseteq \mathcal{P}'_1$ is well defined.

Denote by $\mathcal{P}'' = \mathcal{P'}_1 \cup \boldsymbol{\beta}(\mathcal{P'}_1) \subseteq \underline{K}(C' \otimes \mathrm{C}(\mathbb{T}))$, and then denote by N_1 the integers of Lemma 9.6 of [15] with respect to $C' \otimes \mathrm{C}(\mathbb{T})$, $\pi(\mathcal{G}') \otimes \{1, z\}$ (in the place of $\mathcal{G}$), $\delta'/2$ (in the place of δ), and $\mathcal{P}''$ (in the place of $\mathcal{P}$), where z denotes the identity function on $\mathbb{T}$.

Let M (in place of N) be the constant of Theorem 2.1 of [5] with respect to X', $\mathcal{H}' \cup \mathcal{H}''$ and $\gamma_1'/2$. Without loss of generality, one may assume that $M > 8/(N_1\gamma)$.

Set

$$u_i = ((\mathbf{1}_n - p_i) + (p_i \otimes z))((\mathbf{1}_n - q_i) + (q_i \otimes z))^* \;\; i = 1, 2, ..., k. \tag{e 3.44}$$

We may assume that there are unitaries $u_i' \in M_n(C')$ such that $\imath(u_i') = u_i$, $i = 1, 2, ..., k$.

Choose unitaries $v_i \in U_0(A)$ such that $\Gamma(x_i) = \overline{v_i}$ for each $1 \le i \le k$, and choose $T > 0$ such that $\mathrm{cel}(v_i) < T$, $\quad i = 1, ..., k$.

Also write $K_1(C') = \mathbb{Z}^t \oplus \mathrm{Tor}(K_1(C'))$ and $K_0(C') = \mathbb{Z}^{k'} \oplus \mathrm{Tor}(K_0(C'))$, and let

$$\{y_1 = [e_1], ..., y_{r'} = [e_{r'}], y_{r'+1}, ..., y_{k'}\}$$

be the standard generators of $\mathbb{Z}^{k'}$ with $y_i \in \ker \rho_{C'}$, $i = r'+1, ..., k'$, and e_i, $i = 1, ..., r'$, projections.

By choosing a larger $\mathcal{G}''$ and a smaller δ', we may assume that, for any unital $\mathcal{G}'' \cup \{1, z\}$-$\delta'$-multiplicative contractive completely positive linear map L' from C' to an arbitrary C*-algebra induces a well-defined homomorphism on $s_1(K_1(C' \otimes C(\mathbb{T})))$.

Since $TR(A) \le 1$, there is an interval algebra $I \subset A$ with $p = 1_I$ and $\mathcal{G}''$-$\delta'/4$-multiplicative completely positive linear maps $L_0 : C' \to (1-p)A(1-p)$ and $L_1 : C' \to I$ such that

(1) $\|(L_0(\pi(f)) + L_1(\pi(f))) - \varphi(f)\| < \min\{\epsilon/8, \delta'/16, \gamma_1/4\}$, for any $f \in \mathcal{F} \cup \mathcal{H}' \cup \mathcal{H}''$,

(2) $[\varphi]|_{\mathcal{P}'} = [L_0 \circ \pi]|_{\mathcal{P}'} + [L_1 \circ \pi]|_{\mathcal{P}'}$,

(3) $I = \bigoplus_i M_{n_i}(\mathrm{C}([0, 1]))$ with $n_i > \max\{16(\dim(X) + 1)N_1/\gamma_1, 2M - 2N_1(\dim(X) + 1)\}$,

(4) there are unitaries $v_i' \in (1-p)A(1-p)$ and $v_i'' \in I$ such that $\mathrm{cel}(v_i' \oplus p) < \gamma/4$ in A (by Lemma 3.12) and
$$\|v_i - (v_i' + v_i'')\| < \gamma/4, \quad i = 1, ..., k,$$

(5) moreover, by applying 2.21 of [15], one may assume that for any $r' + 1 \le i \le k'$
$$|\tau(L_1(y_i))| < \gamma_1/8N_2, \quad \forall \tau \in \mathrm{T}(I).$$

There is a subgroup $G_0 \subset \mathbb{Z}^{k'} \subset K_0(C')$ such that $G_0 \cong \mathbb{Z}^k$ and generators $\{g_1, g_2, ..., g_k\} \subset G_0$ such that $\imath_{*0}(g_i) = x_i$, $i = 1, 2, ..., k$. Without loss of generality, we may assume that $[u_i'] = g_i$, $i = 1, 2, ..., k$. Define a homomorphism $\Gamma_1 : K_0(C') \to U_0(I)/CU(I)$ as follows: First define $\Gamma_1(g_i) = \overline{v_i''}$, $i = 1, 2, ..., k$. This gives a homomorphism from $G_0 \to U_0(I)/CU(I)$. Since $U_0(I)/CU(I)$ is divisible, it extends to a homomorphism Γ_1 from $K_0(C')$ to $U_0(I)/CU(I)$. Note that since $U_0(I)/CU(I)$ is also torsion free, $\Gamma_1|_{\mathrm{Tor}(K_0(C'))} = 0$.

Denote by $m_i = n_i\gamma_1/8 + 2N_1(\dim(X) + 1)$. Note that

$$n_i - m_i > M \quad \text{and} \quad m_i/n_i < \gamma_1/4.$$

By Theorem 2.1 of [5], there is a homomorphism

$$\Psi : C' \to \bigoplus_i M_{n_i - m_i}(\mathrm{C}([0,1])) \subseteq I$$

such that

$$|\tau \circ \Psi(h) - \tau \circ L_1(\pi(h))| < \gamma_1/2, \quad \forall h \in \mathcal{H} \cup \mathcal{H}',\ \forall \tau \in \mathrm{T}(I).$$

Define

$$\kappa = ([L_1] - [\Psi]) \oplus 0 \in \mathrm{Hom}_\Lambda(\underline{K}(C' \otimes \mathrm{C}(\mathbb{T})), \underline{K}(A)),$$

where $\underline{K}(C' \otimes \mathrm{C}(\mathbb{T}))$ is identified as $\underline{K}(C') \oplus \boldsymbol{\beta}(\underline{K}(C'))$.

Note that $K_1(C' \otimes \mathrm{C}(\mathbb{T})) \cong K_1(C') \oplus K_0(C')$. It may also be written as $\mathbb{Z}^t \oplus \mathbb{Z}^{k'} \oplus \mathrm{Tor}(K_1(C' \otimes \mathrm{C}(\mathbb{T})))$, where k' is the rank of of $K_0(C')$.

Define a map $\lambda : \mathbb{Z}^t \oplus Z^{k'} \to U_0(I)/CU(I)$ as follows:

$$\lambda(x) = L^\ddagger \circ s_1(x)(\Psi^\ddagger(x^*)) \ \text{ for all } \ x \in K_1(C') \ \text{ and} \tag{e 3.45}$$

$$\lambda|_{\mathbb{Z}^{k'}} = \Gamma_1|_{\mathbb{Z}^{k'}}. \tag{e 3.46}$$

Note that for any $\tau \in \mathrm{T}(I)$ and any $i = r'+1, ..., k'$, one has that

$$|\tau(\kappa(y_i))| = |\tau(L_1(y_i)) - \tau(\Psi(y_i))| = |\tau(L_1(y_i))| < \delta.$$

By Lemma 9.6 of [15], there is a $\mathcal{G}'' \otimes \{1, z\}$-$\delta'/4$-multiplicative map

$$\Phi : C' \otimes \mathrm{C}(\mathbb{T}) \to \bigoplus_i M_{m_i}(\mathrm{C}([0,1]))$$

such that

$$[\Phi] = \kappa \quad \text{and} \quad \Phi^\ddagger|_{s_1(\mathbb{Z}^t \oplus \mathbb{Z}^{k'})} = \lambda.$$

Denote by

$$w' = (1-p) \oplus \langle \Phi(1 \oplus z) \rangle \oplus (\bigoplus_i 1_{M_{n_i - m_i}})$$

and $\psi : C' \to A$ by

$$\psi = L_0 \oplus \Phi|_{C' \otimes 1} \oplus \Psi.$$

Since Φ is $\mathcal{G}'' \otimes \{1, z\}$-$\delta'/4$-multiplicative, it is clear that

$$\|[\psi(\pi(f)), w']\| < \epsilon/4$$

and

$$\mathrm{Bott}(\psi \circ \pi, w') = \kappa \circ \beta \circ \pi = 0.$$

Moreover,

$$\begin{aligned}
\mathrm{dist}(\overline{\langle \psi(u_i') \rangle}, \Gamma(x_i)) \quad &\approx_{\gamma/4} \quad \mathrm{dist}(\overline{(1-p) \oplus \langle \Phi(u_i') \rangle \oplus (\bigoplus_i 1_{M_{n_i - m}})}, \overline{v_i' \oplus v_i''}) \\
&= \quad \mathrm{dist}(\overline{(1-p) \oplus \Gamma_1([u_i']) \oplus (\bigoplus_i 1_{M_{n_i - m}})}, \overline{v_i' \oplus v_i''}) \\
&= \quad \mathrm{dist}(\overline{1 - p \oplus v_i''}, \overline{v_i' \oplus v_i''}) \\
&= \quad \mathrm{dist}(\overline{1}, \overline{v_i' \oplus p}) \approx_{\gamma/4} 0.
\end{aligned}$$

On the other hand, the map $\psi\circ\pi$ is $\mathcal{G}'$-δ'-multiplicative, and

$$[\psi\circ\pi]|_{\mathcal{P}'} = [L_0\circ\pi]|_{\mathcal{P}'} + [\Psi|_{C\otimes 1}\circ\pi]|_{\mathcal{P}'} + [\Phi\circ\pi]|_{\mathcal{P}'} = [L_0\circ\pi]|_{\mathcal{P}'} + [L_1\circ\pi]|_{\mathcal{P}'} = [\varphi]|_{\mathcal{P}'}.$$

One also has that, for any $u\in\mathcal{U}'$,

$$\begin{array}{ll} & \mathrm{dist}(\overline{\varphi(u)}, \overline{\langle\psi(\pi(u))\rangle}) \\ \approx_{\gamma_2} & \mathrm{dist}(\overline{\langle L_0(\pi(u))\oplus L_1(\pi(u))\rangle}, \overline{\langle L_0(\pi(u))\rangle\oplus\langle(L_1(\pi(u))\Psi(\pi(u^*)))\rangle\oplus\Psi(\pi(u))}) \\ = & 0, \end{array}$$

and for any $h\in\mathcal{H}'\cup\mathcal{H}''$,

$$\begin{array}{rll} |\tau(\varphi(h))-\tau(\psi(\pi(h)))| & \approx_{\gamma_1/4} & |\tau(L_0(\pi(h))+L_1(\pi(h)))-\tau(L_0(\pi(h))+\Psi(\pi(h)\otimes 1)+\Phi(\pi(h)))| \\ & \approx_{\gamma_1/2} & \tau(\Psi(\pi(h)\otimes 1))\approx_{\gamma_1/4} 0. \end{array}$$

It then follows from Lemma 3.4 of [14] that

$$\mu_{\tau\circ\psi\circ\pi}(O_r) > \Delta(r/3)/2$$

for any $r>\eta'$. By Theorem 5.3 of [14], there is a unitary v such that

$$\|\varphi(f)-v^*\psi(\pi(f))v\| < \epsilon/2, \quad \forall f\in\mathcal{F}.$$

Then the unitary $w := v^*w'v$ satisfies the lemma. □

Corollary 3.14. *The statement of Theorem 3.13 still holds if* $\mathrm{C}(X)$ *is replaced by* $PM_n(\mathrm{C}(Y))P$ *for a compact subset* Y *of a finite CW-complex and a projection* P *in* $M_n(\mathrm{C}(Y))$*.*

Proof. The corollary clearly holds for $C = M_n(C(X))$ (in this case, X is even not required to be finite dimensional). In what follows we will use this case of the corollary to prove the general case.

Assume that $C = PM_n(C(X))P$. As in the proof of 3.10, without loss of generality, we may assume that $P(x)\neq 0$ for all $x\in X$. Since X is a compact subset of a finite CW-complex, there is an integer d and a projection $Q\in M_d(PM_n(C(X))P)$ such that

$$QM_d(PM_n(C(X))P)Q\cong M_r(C(X))$$

for some integer r. Note that $Q(x)\neq 0$ for all $x\in X$.

Without loss of generality, one may assume that $P\preceq Q$, that is, there is also a partial isometry $V\in M_d(PM_n(C(X))P)$ such that $VV^*\leq Q$ and $V^*V=\{P,0,...,0\}$. In particular, V induces an isomorphism between $PM_n(C(X))P$ and the unital hereditary subalgebra of $QM_d(PM_n(C(X))P)Q$ generated by VV^*.

Fix d, Q, and V.

Since X is compact, there is an integer $l\geq 1$ such that $X = X_1\sqcup\cdots\sqcup X_l$ such that the ranks of the restrictions of P and Q to each X_i, $1\leq j\leq l$, are constant. Denote by P_j and Q_j the restriction of P and Q to X_j respectively. Let $R=\max_{1\leq j\leq 1}\{\mathrm{rank}Q_j\}$.

Let $G\subseteq K_0(C)$ be a finitely generated group with a fixed decomposition $G=\mathbb{Z}^k\oplus\mathrm{Tor}(G)$ with $\mathbb{Z}^k$ generated by

$$\{x_1=[p_1]-[q_1], x_2=[p_2]-[q_2], ..., x_k=[p_k]-[q_k]\},$$

where $p_i, q_i\in M_m(C)$ (for some integer $m\geq 1$) are projections, $i=1,...,k$.

Let A be a unital simple C*-algebra with $TR(A) \le 1$, and let $\varphi : C \to A$ be a monomorphism. Let $\mathcal{F} \subseteq C$, $\mathcal{P} \subseteq \underline{K}(C)$, $\epsilon > 0$, $\gamma > 0$, and $\Gamma : \mathbb{Z}^k \to U_0(A)/CU(A)$ be a homomorphism.

Denote by $q = (\varphi \otimes 1_{M_d})(Q)$, $e = \varphi \otimes 1_{M_d}(VV^*) \in qM_d(A)q$ and $v = \varphi \otimes 1_{M_d}(V) \in qM_d(A)q$.

By Remark 2.3, one may choose unitaries $v_1, ..., v_k \in U_0(M_m(A))$ such that

$$\Gamma(x_i) = \overline{v_i} \in U_0(M_m(A))/CU(M_m(A)), \quad i = 1, ..., k.$$

Then the elements $(v \otimes 1_m)v_i(v \otimes 1_m)^*$, $i = 1, ..., k$, are unitaries in $M_m(eM_d(A)e) = (e \otimes 1_m)(M_m(qM_d(A)q))(e \otimes 1_m)$.

Choose $\epsilon_1 > 0$ and a finite subset $\mathcal{G}_1 \subseteq QM_d(C)Q$ such that $VV^* \in \mathcal{G}_1$ and if there is a unitary $u \in qM_d(A)q$ with

$$\|[(\varphi \otimes 1_{M_d})|_{QM_d(C)Q}(c), u]\| < \epsilon_1, \quad \forall c \in \mathcal{G}_1,$$

then

$$\|[(\varphi \otimes 1_{M_d})|_{VCV^*}(VcV^*), u]\| < \epsilon, \quad \forall c \in \mathcal{G}.$$

By choosing ϵ_1 sufficiently small (note that $VV^* \in \mathcal{G}_1$), the element v^*uv can be assumed to be invertible in A and

$$\|[\varphi(c), \langle v^*uv \rangle]\| < \epsilon, \quad \forall c \in \mathcal{G}. \tag{e 3.47}$$

Using the same argument as that of Corollary 3.10, one may choose a finite subset $\mathcal{G}_2 \subseteq QM_d(C)Q$ and $\epsilon_2 > 0$ such that if

$$\|[(\varphi \otimes 1_{M_d})|_{QM_d(C)Q}(c), u]\| < \epsilon_2, \quad \forall c \in \mathcal{G}_2,$$

and

$$\mathrm{Bott}(\varphi \otimes 1_{M_d}|_{QM_d(C)Q}, u)|_{V\mathcal{P}V^*} = 0,$$

then

$$\mathrm{Bott}(\varphi \otimes 1_{M_d}|_{VCV^*}, \langle eue \rangle)|_{V\mathcal{P}V^*} = 0.$$

Then one may assume further that ϵ_2 is sufficiently small so that $\|v^*\langle eue \rangle v - \langle v^*uv \rangle\|$ is small enough so that

$$\mathrm{Bott}(\varphi, \langle v^*uv \rangle)|_{\mathcal{P}} = 0. \tag{e 3.48}$$

Denote by $\bar{V} = V \otimes 1_m$ and $\bar{v} = v \otimes 1_m$. Note that $\bar{V}p_i\bar{V}^*, \bar{V}q_i\bar{V}^* \in M_m(QM_d(C)Q)$.

Define

$$\Gamma' : \mathbb{Z}^k \to U_0(M_m(qM_d(A)q))/CU(M_m(qM_d(A)q))$$

by

$$\Gamma'(x_i) = \overline{\bar{v}v_i\bar{v}^* + (q \otimes 1_m - e \otimes 1_m)}, \quad i = 1, ..., k.$$

One may choose a finite subset $\mathcal{G}_3 \subseteq QM_d(C)Q$ and $\epsilon_3 > 0$ such that if there is a unitary $u \in qM_d(A)q$ such that

$$\|[(\varphi \otimes 1_d)|_{QM_d(C)Q}(c), u]\| < \epsilon_3, \quad \forall c \in \mathcal{G}_3,$$

and if

$$\begin{aligned} & \mathrm{dist}(\overline{\langle (q \otimes 1_m - \varphi(\bar{V}p_i\bar{V}^*) + \varphi(\bar{V}p_i\bar{V}^*)u \otimes 1_m)(q \otimes 1_m - \varphi(\bar{V}q_i\bar{V}^*) + \varphi(\bar{V}q_i\bar{V}^*)u^* \otimes 1_m \rangle}, \Gamma'(x_i)) \\ < \quad & \frac{\gamma}{(R + \frac{1}{8})} \end{aligned}$$

for any $1 \le i \le k$, then

$$\mathrm{dist}(g_i, CU(M_m(qM_d(A)q))) < \frac{\gamma}{2(R + \frac{1}{8})}, \quad i = 1, ..., k, \tag{e 3.49}$$

where

$$g_i = \langle (q\otimes 1_m - \varphi(\bar V p_i \bar V^*) + \varphi(\bar V p_i \bar V^*) u \otimes 1_m)(q\otimes 1_m - \varphi(\bar V q_i \bar V^*) + \varphi(\bar V q_i \bar V^*) u^* \otimes 1_m\rangle (\bar v v_i^* \bar v^* + (q\otimes 1_m - e\otimes 1_m)).$$

One may assume that ϵ_3 is sufficiently small so that

$$\|g_i - (g_i' + (q\otimes 1_m - e\otimes 1_m))\| < \frac{\gamma}{2(R+\frac{1}{8})}, \quad i = 1, ..., k, \tag{e 3.50}$$

where

$$g_i' = \langle (e\otimes 1_m - \varphi(\bar V p_i \bar V^*) + \varphi(\bar V p_i \bar V^*)\langle eue\rangle \otimes 1_m)(e\otimes 1_m - \varphi(\bar V q_i \bar V^*) + \varphi(\bar V q_i \bar V^*)\langle eue\rangle^* \otimes 1_m\rangle (\bar v v_i^* \bar v^*).$$

As in the proof of 3.10, since A is a unital simple C*-algebra with $TR(A) \le 1$, one has that $g' \in U_0(M_m(eM_d(A)e))$. Note that for any $\tau \in T(M_m(qM_d(A)q))$, one has

$$\tau(e\otimes 1_m) \ge \frac{1}{R},$$

and therefore $R[e\otimes 1_m] \ge [q\otimes 1_m - e\otimes 1_m]$. By Lemma 3.3 of [11], one has that

$$\mathrm{dist}(g_i', CU(M_m(eM_d(A)e))) < (R+\frac{1}{8})\frac{\gamma}{(R+\frac{1}{8})} = \gamma, \quad i = 1, ..., k.$$

Then one may also assume further that ϵ_3 is sufficiently small so that

$$\mathrm{dist}(\langle (1_m - \varphi(p_i) + \varphi(p_i)\langle v^*uv\rangle \otimes 1_m)(1_m - \varphi(q_i) + \varphi(q_i)\langle v^*uv\rangle^* \otimes 1_m\rangle v_i^*, CU(M_m(A)) < \gamma, \tag{e 3.51}$$

for any $1 \le i \le k$. That is,

$$\mathrm{dist}(\overline{\langle (1_m - \varphi(p_i) + \varphi(p_i)\langle v^*uv\rangle \otimes 1_m)(1_m - \varphi(q_i) + \varphi(q_i)\langle v^*uv\rangle^* \otimes 1_m\rangle}, \Gamma(x_i)) < \gamma, \tag{e 3.52}$$

for any $1 \le i \le k$,

Now, since $Q(M_d(PM_n(C(X))P))Q \cong M_r(C(X))$, applying the corollary to $M_r(C(X))$, one obtains a unitary $u \in qM_d(A)q$ such that

$$\|[(\varphi\otimes 1_{M_d})|_{QM_d(C)Q}(c), u]\| < \min\{\epsilon_1, \epsilon_2, \epsilon_3\}, \quad \forall c \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3,$$

$$\mathrm{Bott}((\varphi\otimes 1_{M_d})|_{QM_d(C)Q}, u)|_{V\mathcal{P}V^*} = 0,$$

and

$$\begin{aligned} &\mathrm{dist}(\overline{\langle (q\otimes 1_m - \varphi(\bar V p_i \bar V^*) + \varphi(\bar V p_i \bar V^*) u \otimes 1_m)(q\otimes 1_m - \varphi(\bar V q_i \bar V^*) + \varphi(\bar V q_i \bar V^*) u^* \otimes 1_m\rangle}, \Gamma'(x_i)) \\ < &\frac{\gamma}{(R+\frac{1}{8})} \end{aligned}$$

for any $1 \le i \le k$,

By (e 3.47), (e 3.48), and (e 3.52), the unitary $w = \langle v^*uv\rangle \in A$ satisfies the corollary. □

Lemma 3.15. *Let $C = \mathrm{C}(X)$ with X a compact metric space, and let A be a simple C*-algebra with $TR(A) \le 1$. Suppose that $h : C \to A$ is a unital homomorphism, and $\varphi : C \to A$ is a non-unital homomorphism with*

$$h_{*1} = \varphi_{*1}.$$

Denote by $p = \varphi(1_C)$. For any $\epsilon > 0$, any finite subset $\mathcal{F} \subseteq C$, any finite subset $\mathcal{P} \subseteq \underline{K}(C)$, and any finite subset $\mathcal{U} \subseteq U_c(C)$, there is a $\mathcal{F}$-ϵ-multiplicative map $\Phi : C \to (1-p)A(1-p)$ such that

$$[\Phi]|_{\mathcal{P}} = [H]|_{\mathcal{P}},$$

where $H : C \to pAp$ is the direct sum of finitely many point-evaluations and

$$\mathrm{dist}(h^{\ddagger}(\overline{u})^{-1}\overline{\langle (\Phi\oplus\varphi)(u)\rangle}, \overline{1}) < \epsilon, \quad \forall u \in \mathcal{U}.$$

Proof. Since C can be written as an inductive limit of the C*-algebras of continuous functions on finite CW-complexes, without loss of generality, one may assume that X is a finite CW-complex.

Denote by N_1 and N_2 the constants of Lemma 9.6 of [15] with respect to X, $\mathcal{F}$ (in the place of $\mathcal{G}$), ϵ (in the place of δ).

Choose $\mathcal{G} \subseteq C$ and $\delta > 0$ such that for any C*-algebra B and any $\mathcal{G}$-δ-multiplicative map $L : C \to B$, the element $L(u)$ is invertible for all $u \in \mathcal{U}$.

Since $h_{*1} = \varphi_{*1}$ and A has stable rank one, there is $T > 0$ such that

$$\mathrm{cel}(h(u)((1-p) \oplus \varphi(u^*))) < T, \quad \forall u \in \mathcal{U}.$$

Since $TR(A) \leq 1$, there is a interval algebra $I \subseteq A$ with $q = 1_I$ and $\mathcal{G}$-δ-multiplicative maps $h_0, \varphi_0 : C \to (1-q)A(1-q)$, $h_1, \varphi_1 : C \to I$ such that

(1) $\|h(u) - (h_0(u) \oplus h_1(u))\| < \epsilon/4$ and $\|\varphi(u) - (\varphi_0(u) \oplus \varphi_1(u))\| < \epsilon/4$ for any $u \in \mathcal{U}$.

(2) The element $p_0 := h_0(1_C) - \varphi_0(1_C)$ is a projection in $(1-q)A(1-q)$, and the element $p_1 := h_1(1_C) - \varphi_1(1_C)$ is a projection in I; moreover, $p_0 + p_1 = p$.

(3) $\mathrm{dist}(\overline{\langle h_0(u) \oplus q\rangle}, \overline{1_A}) < \epsilon/4$ and $\mathrm{dist}(\overline{\langle \varphi_0(u) \oplus p_0 \oplus q\rangle}, \overline{1_A}) < \epsilon/4$ for any $u \in \mathcal{U}$.

(4) The rank of p_1 is at least $N_1(\dim(X) + 1)$.

Then it follows from Lemma 9.6 of [15] that there is a $\mathcal{F}$-ϵ-multiplicative map $\Phi' : C \to p_1 I p_1$ such that

$$[\Phi']|_{\mathcal{P}} = [H']|_{\mathcal{P}},$$

where $H' : C \to p_1 I p_1$ is the direct sum of finitely many point evaluation maps and

$$(\Phi')^{\ddagger}(u) = h_1^{\ddagger}(u)\overline{\langle \varphi_1(u^*)\rangle}, \quad \forall u \in \mathcal{U}.$$

Let $\Phi_0 : C \to (1-q)A(1-q)$ be the map $f \to f(\xi)(1-p_0)$ for some $\xi \in X$, and define

$$\Phi = \Phi_0 \oplus \Phi' : \ C \to (p_0 + p_1)A(p_0 + p_1) = pAp.$$

It is clear that

$$[\Phi]|_{\mathcal{P}} = [H]|_{\mathcal{P}},$$

where $H : C \to pAp$ is a direct sum of finitely many point-evaluations. Furthermore, for any $u \in \mathcal{U}$, one has

$$\begin{aligned}
\mathrm{dist}(\overline{\langle \Phi(u)\rangle \oplus \varphi(u)}, \overline{h(u)}) &< \mathrm{dist}(\overline{\langle((\Phi_0(u) \oplus \varphi_0(u)) \oplus (\Phi'(u) \oplus \varphi_1(u))\rangle}, \overline{\langle h_0(u) \oplus h_1(u)\rangle}) + \epsilon/4 \\
&= \mathrm{dist}(\overline{\langle p_0 \oplus \varphi_0(u) \oplus q\rangle}, \overline{1_A}) + \mathrm{dist}(\overline{1_A}, \overline{\langle h_0(u) \oplus q\rangle}) + \\
&\quad\ \mathrm{dist}(\overline{\langle(1-q) \oplus \Phi'(u) \oplus \varphi_1(u)\rangle}, \overline{\langle(1-q) \oplus h_1(u)\rangle}) + \epsilon/4 \\
&\leq \mathrm{dist}(\overline{\langle \Phi'(u) \oplus \varphi_1(u)\rangle}, \overline{\langle h_1(u)\rangle}) + 3\epsilon/4 \\
&= 3\epsilon/4 < \epsilon.
\end{aligned}$$

This proves the lemma. ∎

Theorem 3.16. *Let C be an AH-algebra, and let A be a simple C*-algebra with $TR(A) \leq 1$. Suppose that $h : C \to A$ is a monomorphism. Then, for any $\epsilon > 0$, any finite subset $\mathcal{F} \subseteq C$ and any finite subset $\mathcal{P} \subseteq \underline{K}(C)$, there is a C*-algebra $C' \cong PM_n(\mathrm{C}(X'))P$ for some finite CW-complex X' with $K_1(C') = \mathbb{Z}^k \oplus \mathrm{Tor}(K_1(C'))$ and a homomorphism $\iota : C' \to C$ with*

$\mathcal{P} \subseteq [\iota](\underline{K}(C'))$, a finite subset $\mathcal{Q} \subseteq \mathbb{Z}^k \subset K_1(C')$ and $\delta > 0$ satisfying the following: Suppose that $\kappa \in \mathrm{Hom}_\Lambda(\underline{K}(C' \otimes \mathrm{C}(\mathbb{T})), \underline{K}(A))$ with

$$|\rho_A \circ \kappa(\boldsymbol{\beta}(x))(\tau)| < \delta, \quad \forall x \in \mathcal{Q}, \ \forall \tau \in \mathrm{T}(A).$$

Then there exists a unitary $u \in A$ such that

$$\|[h(c), u]\| < \epsilon, \quad \forall c \in \mathcal{F} \quad \textit{and} \quad \mathrm{Bott}(h \circ \iota, u) = \kappa \circ \boldsymbol{\beta}.$$

Moreover, there is a sequence of C-algebras C_n with the form $C_n = P_n M_{r(n)}(C(X_n))P_n$, where each X_n is a finite CW-complex and $P_n \in M_{r(n)}(C(X_n))$ a projection, such that $C = \varinjlim(C_n, \varphi_n)$ for a sequence of unital homomorphisms $\varphi_n : C_n \to C_{n+1}$ and one may choose $C' = C_n$ and $\iota = \varphi_n$ for some integer $n \geq 1$.*

Proof. The proof is similar to that of Theorem 6.3 of [12]. Without loss of generality, one may assume that $C = \mathrm{C}(X)$ for some compact metric space X. Denote by

$$\Delta(a) = \inf\{\mu_{\tau\circ h}(O_a); \ \tau \in \mathrm{T}(A), O_a \text{ is an open ball of } X \text{ with radius } a\}.$$

Since A is simple and $\mathrm{T}(A)$ is compact, $\Delta(a)$ is a nondecreasing function from $(0,1)$ to $(0,1)$.

Let B be a unital separable simple amenable C*-algebra with $TR(B) = 0$ satisfying the UCT and

$$(K_0(B), K_0^+(B), [1_B], K_1(B)) \cong (K_0(A), K_0^+(A), [1_A], K_1(A)).$$

Then there is an embedding $\iota' : B \to A$ such that $[\iota']$ induces an identification of the above. In the following, we identify B as a C*-subalgebra of A.

Let $\epsilon_1 > 0$ with $\epsilon_1 < \epsilon$, and let $\mathcal{F}_1 \supseteq \mathcal{F}$ be a finite subset such that for any unital homomorphism $H : C \to A$ and unitary $u' \in A$ satisfying

$$\|[H(c), u']\| < \epsilon_1, \quad \forall c \in \mathcal{F}_1,$$

the map $\mathrm{Bott}(H, u')|_{\mathcal{P}}$ is well defined; moreover, if $H' : C \to A$ is any other unital monomorphism satisfying

$$\|H(c) - H'(c)\| < \epsilon_1, \quad \forall c \in \mathcal{F}_1,$$

then

$$\mathrm{Bott}(H, u')|_{\mathcal{P}} = \mathrm{Bott}(H', u')|_{\mathcal{P}}.$$

Let η, δ_1 (in the place of δ), $\mathcal{G}_1 \subseteq C$ (in the place of $\mathcal{H}$), $\mathcal{P}' \subseteq \underline{K}(C)$ (in the place of $\mathcal{P}$), and $\mathcal{U} \subseteq U_c(K_1(C))$, γ_1, γ_2 be the constants and finite subsets of Theorem 5.3 of [14] with respect to $\epsilon_1/2$, $\mathcal{F}_1$, and $\Delta(\cdot/3)/2$.

Let δ_2 and $\mathcal{G}_2' \subseteq C$ be the constant and finite subset required by Lemma 3.4 of [14] with respect to Δ, η, and $\lambda_1 = \lambda_2 = 1/2$. Denote by $\mathcal{G}_2 = \mathcal{G}_1 \cup \mathcal{G}_2'$. Without loss of generality, one may assume that $\delta_2 < \gamma_1$.

By Lemma 6.2 of [12], there is a $\mathcal{G}_1$-δ_1-multiplicative map $h_0 : C \to p_0 B p_0$ with $\tau(p_0) < \delta_2/4$ and a unital homomorphism $h_1' : C \to F$, where F is a finite dimensional C*-subalgebra of B with $1_F = 1 - p_0$ such that

$$[h_0 + h_1']|_{\mathcal{P}'} = [h]|_{\mathcal{P}'} \text{ in } KL(C, A).$$

Let $C' \subseteq C$, $1 > \delta_3 > 0$ and $\mathcal{Q}' \subseteq K_1(C')$ (in place of $\mathcal{Q}$) be the constant and finite subset required by Lemma 6.1 of [12] with respect to $\mathcal{F}$, $\mathcal{P}$, and $p_0 B p_0$. We may write $K_1(C') = \mathbb{Z}^k \oplus \mathrm{Tor}(K_1(C'))$. Let $\mathcal{Q} \subset \mathbb{Z}^k$ be a finite subset such that

$$\mathcal{Q}' = \{x + y : x \in \mathcal{Q} \text{ and } y \in \mathrm{Tor}(K_1(C'))\}.$$

Let $\delta = \min\{\delta_3\delta_1/16\pi, \delta_3\delta_2/4\}$. Now let $\kappa \in \mathrm{Hom}_\Lambda(\underline{K}(C' \otimes \mathrm{C}(\mathbb{T})), \underline{K}(A))$ with

$$|\rho_A \circ \kappa(\boldsymbol{\beta}(x))(\tau)| < \delta \text{ for all } x \in \mathcal{Q} \text{ and } \text{ for all } \tau \in \mathrm{T}(A).$$

Note that this implies

$$|\rho_A \circ \kappa(\boldsymbol{\beta}(x))(\tau)| < \delta \text{ for all } x \in \mathcal{Q}'$$

and for all $\tau \in T(A)$. By Lemma 6.1 of [12], there is a unitary $u_0 \in p_0Bp_0$ such that

$$\|[h_0(c), u_0]\| < \epsilon_1/2, \quad \forall c \in \mathcal{F},$$

and

$$\mathrm{Bott}(h_0 \circ \iota, u_0) = \kappa \circ \boldsymbol{\beta}.$$

Put $u = u_0 + (1 - p_0)$. Then there is a nonzero projection $q_0 \in (1 - p_0)A(1 - p_0)$ such that

$$q_0 f = f q_0, \quad \forall f \in F, \quad \text{and} \quad \tau(q_0) < \delta, \quad \forall \tau \in T(A).$$

Define $\psi_0(c) = q_0 h_1'(c)$ and $\psi_0'(c) = (1 - p_0 - q_0)h_1'$ for all $c \in C$. By Lemma 9.5 of [8], there is C*-subalgebra $B_0 \in (1-p_0-q_0)A(1-p_0-q_0)$ with B_0 an interval algebra and a unital homomorphism $h_1 : C \to B_0$ such that $(h_1)_{*0} = (\psi_0')_{*0}$ and

$$|\tau(h_1(f)) - \tau(1 - p_0 - q_0)\tau(h(f))| < \delta_2/4, \quad \forall f \in \mathcal{G}_2.$$

Define $\psi_1 = h_0 + h_1$. By Lemma 3.15, there is $\mathcal{G}'$-δ_1-multiplicative map $\Phi : C \to q_0Aq_0$ with $\Phi_{*0} = (\psi_0)_{*0}$, $[\Phi]|_{\mathcal{P}'} = [H]|_{\mathcal{P}'}$ in $KL(C, A)$ for some point evaluation map $H : C \to pAp$, and

$$\mathrm{dist}(h^{\ddagger}(\overline{u})^{-1}\overline{\langle(\Phi \oplus \psi_1)(u)\rangle}, \overline{1}) < \gamma_2, \quad \forall u \in \mathcal{U}.$$

Define $h_2 = \Phi \oplus \psi_1$. Then $[h_2]|_{\mathcal{P}'} = [h]|_{\mathcal{P}'}$ in $KL(C, A)$ and for any $u \in \mathcal{U}$,

$$\mathrm{dist}(\overline{\langle h_2(u)\rangle}, \overline{h(u)}) = \mathrm{dist}(\overline{\langle \Phi(u) \oplus \psi_1(u)\rangle}, \overline{h(u)}) \approx_{\gamma_2} 0.$$

Moreover, for any $f \in \mathcal{G}_2$ and any $\tau \in T(A)$,

$$\begin{aligned} |\tau(h(f)) - \tau(h_2(f))| &< \delta_2/4 + |\tau(h(f)) - \tau(h_1(f))| \\ &\leq 3\delta_2/4 + |\tau(1 - p_0 - q_0)\tau(h(f)) - \tau(h_1(f))| \\ &< \delta_2 < \gamma_1. \end{aligned}$$

Note that $\mu_{\tau\circ h}(O_a) \geq \Delta(a)$ for any a, by Lemma 3.4 of [14], one has

$$\mu_{\tau \circ h_2}(O_a) \geq \frac{1}{2}\Delta(a/3)$$

for any $a \geq \eta$. Then, by Theorem 5.3 of [14], there is a unitary $U \in A$ such that

$$\mathrm{ad}U \circ h_2 \approx_{\epsilon_1/2} h, \quad \text{on } \mathcal{F}_1.$$

Define $u = U^*(u_0 + (1 - p_0))U$. Then

$$\|[h(c), u]\| < \epsilon_1, \quad \forall c \in \mathcal{F}_1.$$

Moreover, by the choice of ϵ_1, one has

$$\mathrm{Bott}(h \circ \iota, u) = \mathrm{Bott}(h_2 \circ \iota, u_0 + (1 - p_0)) = \mathrm{Bott}(h_0 \circ \iota, u_0) = \kappa \circ \boldsymbol{\beta},$$

as desired. $\square$

4 Asymptotic unitary equivalence

Lemma 4.1. *Let C be a unital AH-algebra and let A be a unital separable simple C*-algebra with $TR(A) \le 1$. Suppose that $\varphi_1, \varphi_2 : C \to A$ are two unital monomorphisms. Suppose that*

(1) $[\varphi_1] = [\varphi_2]$ *in* $KL(C, A)$, $\varphi_1^\ddagger = \varphi_2^\ddagger$, $(\varphi_1)_\sharp = (\varphi_2)_\sharp$,

(2) $R_{\varphi_1,\varphi_2}(K_1(M_{\varphi_1,\varphi_2})) \subseteq \rho_A(K_0(A))$.

Then, for any increasing sequence of finite subsets $(\mathcal{F}_n)$ of C whose union is dense in C, any increasing sequence of finite subsets $(\mathcal{P}_n)$ of $K_1(C)$ with $\bigcup_{n=1}^\infty \mathcal{P}_n = K_1(C)$ and any decreasing sequence of positive number (δ_n) with $\sum_{n=1}^\infty \delta_n < \infty$, there exists a sequence of unitaries (u_n) in $U(A)$ such that

$$\mathrm{ad}(u_n) \circ \varphi_1 \approx_{\delta_n} \varphi_2 \quad \text{on } \mathcal{F}_n,$$

and

$$\rho_A(\mathrm{bott}_1(\varphi_2, u_n^* u_{n+1})(x)) = 0,$$

for all $x \in \mathcal{P}_n$ for all sufficiently large n.

Proof. The proof is a simple modification of the proof of Lemma 7.1 of [12]. In the place of Theorem 6.3 of [12] being used, one uses the second part of Theorem 3.16 instead. ☐

Theorem 4.2. *Let C be a unital AH-algebra and let A be a unital separable simple C*-algebra with $TR(A) \le 1$. Suppose that $\varphi_1, \varphi_2 : C \to A$ are two unital monomorphisms. Then there exists a continuous path of unitaries $\{u(t) : t \in [0, \infty)\} \subseteq A$ such that*

$$\lim_{t\to\infty} \mathrm{ad}(U(t)) \circ \varphi_1(c) = \varphi_2(c) \quad \text{for all } c \in C$$

if and only if

$$[\varphi_1] = [\varphi_2] \text{ in } KK(C, A),\ (\varphi_1)^\ddagger = (\varphi_2)^\ddagger,\ (\varphi_1)_\sharp = (\varphi_2)_\sharp$$

and

$$\overline{R}_{\varphi_1,\varphi_2} = 0.$$

Proof. We only have to show the "if" part.

Let $C = \varinjlim(C_n, \psi_n)$, where C_n is a C*-algebra in the form of $P_n M_{r(n)}(C(X_n))P_n$ with X_n having a finite covering dimension, and $\varphi_n : C_n \to C_{n+1}$ is a unital monomorphism. Let $(\mathcal{F}_n)$ be an increasing sequence of finite subsets of C such that $\bigcup_{n=1}^\infty \mathcal{F}_n$ is dense in C.

For each n and $0 < a < 1$, define

$$\Delta_n(a) = \inf\{\mu_{\tau\circ\varphi_1}(O_a) : O_a \text{ an open ball of } X_n \text{ with radius } a\}.$$

Since A is simple, one has that $\Delta_n(a) \in (0, 1)$ for any $a \in (0, 1)$.

Consider the mapping torus

$$M_{\varphi_1,\varphi_2} = \{f \in C([0,1], A) : f(0) = \varphi_1(a) \text{ and } f(1) = \varphi_2(a) \text{ for some } a \in C\}.$$

Since C satisfies the Universal Coefficient Theorem, the assumption of $[\varphi_1] = [\varphi_2]$ in $KK(C, A)$ implies the following short exact sequence splits:

$$0 \to \underline{K}(SA) \to \underline{K}(M_{\varphi_1,\varphi_2}) \to^{\pi_0} \underline{K}(C) \to 0.$$

Denote by $\theta : \underline{K}(C) \to \underline{K}(M_{\varphi_1,\varphi_2})$ the splitting map.

Since $\tau \circ \varphi_1 = \tau \circ \varphi_2$ for all $\tau \in T(A)$ and $\overline{R}_{\varphi_1, \varphi_2} = 0$, we may also assume that

$$R_{\varphi_1,\varphi_2}(\theta(x)) = 0,$$

for all $x \in K_1(C)$.

In what follows, for any C*-algebras C'' and A and a homomorphism $\varphi : C'' \to A$, for any $x = [p] - [q] \in K_0(C)$ with projections $p, q \in M_n(A)$ (for some integer $n \geq 1$) and a unitary $u \in A$ with $\|[\varphi(p), \bar{u}]\| < 1/4$ and $\|[\varphi(q), \bar{u}]\| < 1/4$, where $\bar{u} = \mathrm{diag}(u, ..., u)$, define

$$g^{\varphi}_{x,u} := \overline{\langle (1_n - \varphi(p) + \varphi(p)\bar{u})(1_n - \varphi(q) + \varphi(q)\bar{u}^*)\rangle} \in U_n(A)/CU_n(A). \tag{e 4.53}$$

Let $\delta'_n > 0$(in place of δ), η'_n (in place of η), γ'_n (in place of γ), $\mathcal{G}'_n \subseteq C_n$ (in place of $\mathcal{G}$), $\mathcal{P}'_n \subseteq \underline{K}(C_n)$ and $\mathcal{Q}'_n = \{x_{n,1}, ..., x_{n,m(n)}\} \subseteq K_0(C_n)$ (in place of $\mathcal{Q}$) be the constants and finite subsets corresponding to $1/2^{n+1}$, $\mathcal{F}_n$ and Δ_n required by Theorem 3.9. Without loss of generality, one may assume that $[\psi_{n,n+1}](P'_n) \subseteq \mathcal{P}'_{n+1}$ for all n. Note that $\{x_{n,1}, ..., x_{n,m(n)}\}$ are free (hence generate a group $\mathbb{Z}^{m(n)} \subseteq K_0(C_n)$), and write $x_{n,j} = [p_{n,j}] - [q_{n,j}]$ for some projections $p_{n,j}, q_{n,j} \in M_{l(n)}(C_n)$.

Consider the image $[\psi_{n,n+1}](\mathbb{Z}^{m(n)})$, and fix a decomposition

$$[\psi_{n,n+1}](\mathbb{Z}^{m(n)}) = \mathbb{Z}^{k(n)} \oplus \mathrm{Tor}([\psi_{n,n+1}](\mathbb{Z}^{m(n)}))$$

for some integer $k(n)$. One also fixes a lifting of $\mathbb{Z}^{k(n)}$ in $\mathbb{Z}^{m(n)}$. Write $\{y_{n,1}, y_{n,2}, ..., y_{n,k(n)}\}$ a set of generators of $\mathbb{Z}^{k(n)}$, and $\{y'_{n,1}, y'_{n,2}, ..., y'_{n,k(n)}\}$ the corresponding elements in $\mathbb{Z}^{m(n)}$. Note that there are integers $c^{(n)}_{i,j}$ such that

$$x_{n,i} = \sum_{j=1}^{k(n)} c^{(n)}_{i,j} y'_{n,j} + r_i, \quad i = 1, ..., m(n)$$

with $[\psi_{n,n+1}](r_i)$ a torsion element in $K_0(C_{n+1})$.

Therefore, without loss of generality, one may assume that δ'_n is sufficiently small and $\mathcal{G}'_n$ is sufficiently large such that if $h' : C \to A$ is a homomorphism and $u' \in A$ a unitary with $\|[h'(a), u']\| < \delta'_n$ for all $a \in \mathcal{G}'_n$, then

$$\mathrm{dist}(g^h_{x_{n,i},u'}, \sum_{j=1}^{k(n)} c^{(n)}_{i,j} g^h_{y'_{n,j},u'}) < \gamma'_n/8, \quad i = 1, ..., m(n). \tag{e 4.54}$$

We also assumes that $\mathrm{Bott}(h', u')|_{\mathcal{P}_n}$ is well defined whenever $\|[h'(a), u']\| < \delta'_n$ for all $a \in \mathcal{G}'_n$ for any homomorphism h' and unitary u', and moreover, if $h \approx_{\delta'_n} h'$ on $\mathcal{G}'_n$, then

$$\mathrm{Bott}(h, u)|_{\mathcal{P}_n} = \mathrm{Bott}(h', u)|_{\mathcal{P}_n}.$$

Let C'_n (in place of C') with $K_1(C'_n) = \mathbb{Z}^{r(n)} \oplus \mathrm{Tor}(K_1(C'_n))$, $\iota_n : C'_n \to C_n$, $\mathcal{Q}''_n \subseteq \mathbb{Z}^{r(n)}$ (in place of $\mathcal{Q}$), and η_n (in place of δ) be required by Theorem 3.16 for $\mathcal{G}'_n$ (in place of $\mathcal{F}$), $\mathcal{P}'_n$ (in place of $\mathcal{P}$) and $\delta'_n/4$ (in place of ϵ). One also fixes a finite set of generators of $K_1(C'_n)$ for each n. Without loss of generality, one may assume that $\mathcal{Q}''_n$ is the set of standard generators of $\mathbb{Z}^{r(n)}$.

Put $\delta_n = \min\{\eta_n, \delta'_n/2\}$.

By Lemma 4.1, there are unitaries $v_n \in U(A)$ such that

$$\mathrm{ad}(v_n) \circ \varphi_1 \approx_{\delta_{n+1}/4} \varphi_2 \quad \text{on } \psi_{n+1,\infty}(\mathcal{G}'_{n+1}),$$

$$\rho_A(\mathrm{bott}_1(\varphi_2 \circ \iota_n, v_n^* v_{n+1}))(x) = 0 \quad \text{for all } x \in \psi_{n+1,\infty}(K_1(C'_{n+1})),$$

and

$$\|[\varphi_2(a), v_n^* v_{n+1}]\| < \delta_{n+1}/2 \quad \text{for all } a \in \psi_{n+1,\infty}(\mathcal{G}'_{n+1}).$$

Then we have that

$$\mathrm{Bott}(\varphi_1 \circ \iota_{n+1}, v_{n+1} v_n^*) = \mathrm{Bott}(v_n^*(\varphi_1 \circ \iota_{n+1}) v_n, v_n^*(v_{n+1} v_n^*) v_n) = \mathrm{Bott}(\varphi_2 \circ \iota_{n+1}, v_n^* v_{n+1}).$$

In particular, for any $x \in (\psi_{n+1,\infty} \circ \iota_{n+1})_{*1}(K_1(C'_{n+1}))$, one has

$$\mathrm{bott}_1(v_n^* \varphi_1 v_n, v_n^* v_{n+1})(x) = \mathrm{bott}_1(\varphi_2, v_n^* v_{n+1})(x).$$

By applying 10.4 and 10.5 of [9], without loss of generality, we may assume that $\varphi_1 \circ \psi_{n+1,\infty} \circ \iota_{n+1}$ and v_n define an element $\gamma_n \in \mathrm{Hom}_\Lambda(\underline{K}(C'_{n+1}), \underline{K}(M_{\varphi_1 \circ \iota_{n+1}, \varphi_2 \circ \iota_{n+1}}))$ and $[\pi_0] \circ \gamma_n = [\iota_{n+1}]$. Moreover, γ_n factors through $H_n := [\psi_{n+1,\infty} \circ \iota_{n+1}](\underline{K}(C'_{n+1}))$. Thus, one may also regard γ_n being defined on H_n.

Furthermore, by 10.4 and 10.5 of [9], without loss of generality, we may assume that

$$\tau(\log((\varphi_2 \circ \psi_{n+1,\infty} \circ \iota_{n+1}(z_j^*)) \overline{v}_n^* (\varphi_1 \circ \psi_{n+1,\infty} \circ \iota_{n+1}(z_j)) \overline{v}_n)) < \delta_{n+1} \tag{e 4.55}$$

where $\{z_1, ..., z_{r(n)}\} \subseteq U(M_k(C'_{n+1}))$ induces a set of standard generators of $\mathbb{Z}^{r(n)} \subseteq K_1(C'_{n+1})$ and $\overline{v}_n = \mathrm{diag}(\underbrace{v_n, ..., v_n}_{k})$.

Since $\bigcup_{n=1}^\infty [\psi_{n+1,\infty} \circ \iota_{n+1}](\underline{K}(C'_n)) = \underline{K}(C)$ and $[\pi_0] \circ \gamma_n = [\iota_{n+1}]$, one concludes

$$\underline{K}(M_{\varphi_1, \varphi_2}) = \underline{K}(SA) + \bigcup_{n=1}^\infty \gamma_n(H_n). \tag{e 4.56}$$

By passing to a subsequence, one may assume that

$$\gamma_n(H_n) \subseteq \underline{K}(SA) + \gamma_{n+1}(H_{n+1}), \quad n = 1, 2, ...$$

By 10.6 of [9], $\Gamma(\mathrm{Bott}(\varphi_1, v_n v_{n+1}^*))|_{H_n} = (\gamma_n - \gamma_{n+1} \circ [\psi_{n+1}])|_{H_n}$ defines a homomorphism $\xi_n : H_n \to \underline{K}(SA)$. Then define a map $j_n : \underline{K}(SA) \oplus H_n \to \underline{K}(SA) \oplus H_{n+1}$ by

$$(x, y) \mapsto (x + \xi_n(y), [\psi_{n+1}](y)).$$

By (e 4.56), the limit is $\underline{K}(M_{\varphi_1, \varphi_2})$. One has the following diagram

$$\begin{array}{ccccccc}
0 \to & \underline{K}(SA) & \to \underline{K}(SA) & & \bigoplus H_n & \to H_n & \to 0 \\
 & \downarrow_= & \downarrow_= & \swarrow_{\xi_n} & \downarrow_{[\psi_{n+1}]} & \downarrow_{[\psi_{n+1}]} & \\
0 \to & \underline{K}(SA) & \to \underline{K}(SA) & & \bigoplus H_{n+1} & \to H_{n+1} & \to 0.
\end{array}$$

By the assumption that $\overline{R}_{\varphi_1, \varphi_2} = 0$, the map θ also induces the following

$$\ker R_{\varphi_1, \varphi_2} = \ker \rho_A \oplus K_1(C).$$

Define $\zeta_n = \gamma_{n+1}|_{H_n}$, $\theta_n = \theta \circ [\psi_{n+1,\infty}]$, and $\kappa_n = \zeta_n - \theta_n$. Note that

$$\theta_n = \theta_{n+1} \circ [\psi_{n+2}]$$

and

$$\zeta_n - \zeta_{n+1} \circ [\psi_{n+2}] = \xi_n.$$

Since $[\pi_0]\circ(\zeta_n-\theta_n)=0$, κ_n maps H_n into $\underline{K}(SA)$. It follows that

$$\kappa_n-\kappa_{n+1} = \zeta_n-\theta_n-\zeta_{n+1}\circ[\psi_{n+2}]+\theta_{n+1}\circ[\psi_{n+2}] \qquad \text{(e 4.57)}$$
$$= \zeta_n-\zeta_{n+1}\circ[\psi_{n+2}]=\xi_n. \qquad \text{(e 4.58)}$$

It follows from 10.3 of [9] that there are integers $N_1\ge 1$, a δ_{n+1}-$\psi_{n+1}(\mathcal{G}'_{n+1})$-multiplicative map

$$L_n:\psi_{n+1,\infty}\circ\iota_{n+1}(C'_{n+1})\to M_{1+N_1}(M_{\psi_1,\psi_2}),$$

a unital homomorphism $h_0:\psi_{n+1,\infty}\circ\iota_{n+1}(C'_{n+1})\to M_{N_1}(C)$, and a continuous path of unitaries $\{V_n(t):t\in[0,3/4]\}$ of $M_{1+N_1}(A)$ such that $[L_n]|_{\mathcal{P}'_{n+1}}$ is well defined, $V_n(0)=1_{M_{1+N_1}(A)}$,

$$[L_n\circ\psi]|_{\mathcal{P}'_n}=(\theta\circ[\psi_{n+1,\infty}]+[h_0\circ\psi_{n+1,\infty}])|_{\mathcal{P}'_n},$$

$$\pi_t\circ L_n\circ\psi_{n+1,\infty}\approx_{\delta_{n+1}/4}\mathrm{ad}V_n(t)\circ((\varphi_1\circ\psi_{n+1,\infty})\oplus(h_0\circ\psi_{n+1,\infty}))$$

on $\mathcal{G}_{n+1}$ and $t\in[0,3/4]$, and

$$\pi_t\circ L_n\circ\psi_{n+1,\infty}\approx_{\delta_{n+1}/4}\mathrm{ad}V_n(3/4)\circ((\varphi_1\circ\psi_{n+1,\infty})\oplus(h_0\circ\psi_{n+1,\infty}))$$

on $\mathcal{G}_{n+1}$ and $t\in(3/4,1)$, and

$$\pi_1\circ L_n\circ\psi_{n+1,\infty}\approx_{\delta_{n+1}/4}(\varphi_1\circ\psi_{n+1,\infty})\oplus(h_0\circ\psi_{n+1,\infty})$$

on $\mathcal{G}_{n+1}$. Note that $R_{\varphi_1,\varphi_2}(\theta(x))=0$ for all $x\in(\psi_{n+1,\infty})_{*1}(K_1(C_{n+1}))$. As computed in 10.4 of [9],

$$\tau(\log((\varphi_2(z)\oplus h_0(z))^*V_n^*(3/4)(\varphi_1(z)\oplus h_0(z))V_n(3/4)))=0 \qquad \text{(e 4.59)}$$

for $z=(\psi_{n+1,\infty}\circ\iota_{n+1})_{*1}(y)$, where y in the fixed set of generators of $K_1(C'_{n+1})$ and for all $\tau\in T(A)$.

Define $W'_n=\mathrm{diag}(v_n,1)\in M_{1+N_1}(A)$. Then

$$\mathrm{Bott}((\varphi_1\oplus h_0)\circ\psi_{n+1,\infty}\circ\iota_{n+1},W'_n(V_n(3/4))^*)$$

defines a homomorphism $\tilde\kappa_n\in\mathrm{Hom}_\Lambda(\underline{K}(C'_{n+1}),\underline{K}(SA))$.

By (e 4.55), one has

$$\tau(\log((\varphi_2\oplus h_0)\circ\psi_{n+1,\infty}\circ\iota_{n+1}(z_j)^*\tilde V_n^*(\varphi_1\oplus h_0)\circ\psi_{n+1,\infty}\circ\iota_{n+1}(z_j)\tilde V_n))<\delta_{n+1}$$

for $j=1,2,...,r(n)$, where $\tilde V_n=\mathrm{diag}(\overline{v}_n,1)$. Then, by (e 4.59), one has

$$\rho_A(\tilde\kappa_n(z_j))(\tau)<\delta_{n+1},\quad j=1,2,...,r(n).$$

It then follows from Theorem 3.16 that there is a unitary $w'_n\in U(A)$ such that

$$\|[\varphi_1(a),w'_n]\|<\delta'_{n+1}/4,\quad \forall a\in\psi_{n+1,\infty}(\mathcal{G}_{n+1}),$$

and

$$\mathrm{Bott}(\varphi_1\circ\psi_{n+1,\infty}\circ\iota_{n+1},w'_n)|_{\underline{K}(C'_{n+1})}=-\tilde\kappa_n.$$

Put $w_n=v_n^*w'_nv_n$. One has

$$\mathrm{Bott}(\varphi_2\circ\psi_{n+1,\infty}\circ\iota_{n+1},w_n)|_{\underline{K}(C'_{n+1})}=-\tilde\kappa_n|_{\underline{K}(C'_{n+1})}.$$

It follows from 10.6 of [9] that

$$\Gamma(\mathrm{Bott}(\varphi_1\circ\psi_{n+1,\infty},w'_n))=-\kappa_n\quad\text{and}\quad\Gamma(\mathrm{Bott}(\varphi_1\circ\psi_{n+2,\infty},w'_{n+1}))=-\kappa_{n+1},$$

where Γ is defined in 10.6 of [9]. One also has

$$\Gamma(\mathrm{Bott}(\varphi_1\circ\psi_{n+1,\infty}, v_nv_{n+1}^*))|_{H_n}=\zeta_n-\zeta_{n+1}\circ[\psi_{n+2}]=\xi_n.$$

Then, by (e 4.57), one has

$$-\mathrm{Bott}(\varphi_1\circ\psi_{n+1,\infty}, w_n')+\mathrm{Bott}(\varphi_1\circ\psi_{n+1,\infty}, v_nv_{n+1}^*)+\mathrm{Bott}(\varphi_1\circ\psi_{n+1,\infty}, w_{n+1}')=0.$$

Define $u_n'=v_nw_n^*$, $n=1,2,...$ Then,

$$\mathrm{ad}(u_n')\circ\varphi_1\approx_{\delta_{n+1}'/2}\varphi_2,\quad\forall a\in\psi_{n+1,\infty}(\mathcal{G}_{n+1}),$$

and

$$\begin{aligned}
&\mathrm{Bott}(\varphi_2\circ\psi_{n+1,\infty}, (u_n')^*u_{n+1}')\\
=\;&\mathrm{Bott}(\varphi_2\circ\psi_{n+1,\infty}, w_nv_n^*v_{n+1}w_{n+1}^*)\\
=\;&\mathrm{Bott}(\varphi_2\circ\psi_{n+1,\infty}, w_n)+\mathrm{Bott}(\varphi_2\circ\psi_{n+1,\infty}, v_n^*v_{n+1})+\mathrm{Bott}(\varphi_2\circ\psi_{n+1,\infty}, w_{n+1}^*)\\
=\;&\mathrm{Bott}(\varphi_1\circ\psi_{n+1,\infty}, w_n')-\mathrm{Bott}(\varphi_1\circ\psi_{n+1,\infty}, v_nv_{n+1}^*)-\mathrm{Bott}(\varphi_1\circ\psi_{n+1,\infty}, w_{n+1}')\\
=\;&0.
\end{aligned}$$

In what follows, we shall construct unitaries $\{s_n\}\subseteq A$ such that

$$\|[\varphi_2\circ\psi_{n+1,\infty}(a), s_n]\|<\delta_{n+1}'/2,\quad\forall a\in\mathcal{G}_{n+1}' \qquad (e\,4.60)$$

$$\mathrm{Bott}(\varphi_2\circ\psi_{n+1,\infty}, s_n)|_{\mathcal{P}_{n+1}'}=0, \qquad (e\,4.61)$$

and

$$\mathrm{dist}(g^{\varphi_2\circ\psi_{n+1,\infty}}_{x_{n+1,j},s_n^*s_{n+1}}, g^{\varphi_2\circ\psi_{n+1,\infty}}_{x_{n+1,j},(u_n')^*u_{n+1}'})<\gamma_{n+1}'/2. \qquad (e\,4.62)$$

(Recall that $x_{n,j}=[p_{n,j}]-[q_{n,j}]$, $j=1,...,m(n)$ and $n=1,...,$ and $\{x_{n,1},...,x_{n,m(n)}\}$ is free.)

Put $s_1=1$, and assume that $s_1,...,s_n$ has been constructed. Let us construct s_{n+1}. Define the map $\Xi_n':\mathbb{Z}^{m(n+1)}\to U(A)/CU(A)$ by

$$\Xi_n'(x_{n+1,j})=g^{\varphi_2\circ\psi_{n+1,\infty}}_{x_{n+1,j},s_n(u_n')^*u_{n+1}'},\quad j=1,...,m(n+1)$$

with the map $\varphi_2\circ\psi_{n+1,\infty}$ in the place of φ in (e 4.53).

Recall that there are fixed decomposition $[\psi_{n+1,n+2}](\mathbb{Z}^{m(n+1)})=\mathbb{Z}^{k(n+1)}\oplus\mathrm{Tor}([\psi_{n+1,n+2}](\mathbb{Z}^{m(n+1)}))$ (for some integer $k(n+1)$) and a fixed lifting of $\mathbb{Z}^k$ in $\mathbb{Z}^{m(n+1)}$ for each n. Also recall that $\{y_{n+1,1}, y_{n+1,2},..., y_{n+1,k(n+1)}\}$ is a fixed set of generators for $\mathbb{Z}^{k(n+1)}$, and $\{y_{n+1,1}', y_{n+1,2}',..., y_{n+1,k(n+1)}'\}$ are their liftings in $\mathbb{Z}^{m(n+1)}$. Then define the map $\Xi_n:\mathbb{Z}^{k(n+1)}\to U(A)/CU(A)$ by

$$\Xi_n(y_j)=\Xi_n'(y_j'),\quad j=1,...,k(n+1).$$

Let $\epsilon_n''>0$ be arbitrary (which will be fixed later). Applying Theorem 3.13 to C_{n+2} (in place of C), A, $\mathcal{G}_{n+2}'$ (in place of $\mathcal{F}$), $\mathcal{P}_{n+2}'$ (in place of $\mathcal{P}$), ϵ_n'' (in place of ϵ and in place of γ), and Ξ_n (in place of Γ), there is a unitary $s_{n+1}\in A$ such that

$$\|[\varphi_2\circ\psi_{n+1,\infty}(a), s_{n+1}]\|<\epsilon_n'',\quad\forall a\in\mathcal{G}_{n+1}', \qquad (e\,4.63)$$

$$\mathrm{Bott}(\varphi_2\circ\psi_{n+2,\infty}, s_{n+1})|_{\mathcal{P}_{n+2}'}=0, \qquad (e\,4.64)$$

and

$$\mathrm{dist}(g^{\varphi_2\circ\psi_{n+2,\infty}}_{y_{n+1,j},s_{n+1}}, \Xi_n(y_{n+1,j}))<\epsilon_n'',\quad j=1,...,k(n+1). \qquad (e\,4.65)$$

By choosing $\epsilon''_n < \delta'_{n+1}/2$ sufficiently small, it follows from (e 4.63) that the unitary s_{n+1} satisfies (e 4.60). Since $\pi([\psi_{n+1,n+2}(x_{n+1,j})])$ is in the subgroup generated by $\{y_{n+1,1}, ..., y_{n+1,k(n+1)}\}$, where π is the projection map from $[\psi_{n+1,n+2}](\mathbb{Z}^{m(n+1)})$ to $\mathbb{Z}^{k(n+1)}$, by choosing ϵ''_n smaller, it follows from (e 4.65) and (e 4.54) that

$$\mathrm{dist}(g^{\varphi_2\circ\psi_{n+1,\infty}}_{x_{n+1,j},s_{n+1}}, \Xi'_n(x_{n+1,j})) < \gamma'_{n+1}/4, \quad j = 1, ..., m(n+1),$$

which is

$$\mathrm{dist}(g^{\varphi_2\circ\psi_{n+1,\infty}}_{x_{n+1,j},s_{n+1}}, g^{\varphi_2\circ\psi_{n+1,\infty}}_{x_{n+1,j},s_n(u'_n)^*u'_{n+1}}) < \gamma'_{n+1}/4, \quad j = 1, ..., m(n+1).$$

Hence

$$\mathrm{dist}(g^{\varphi_2\circ\psi_{n+1,\infty}}_{x_{n+1,j},s_n^*s_{n+1}}, g^{\varphi_2\circ\psi_{n+1,\infty}}_{x_{n+1,j},(u'_n)^*u'_{n+1}}) < \gamma'_{n+1}/2, \quad j = 1, ..., m(n+1),$$

which verifies (e 4.62).

Therefore, one obtains the sequence of unitaries (s_n) satisfying (e 4.60), (e 4.61) and (e 4.62).

Define $u_n = u'_n s_n^*$, $n = 1, 2, ...$ Then it follows from (e 4.60) and (e 4.61) that

$$\|[\varphi_2 \circ \psi_{n+1,\infty}, u_n^* u_{n+1}]\| < \delta'_n, \tag{e 4.66}$$

and

$$\mathrm{Bott}(\varphi_2 \circ \psi_{n+1,\infty}, u_n^* u_{n+1})|_{\mathcal{P}'_{n+1}} = 0. \tag{e 4.67}$$

It also follows from (e 4.62) that

$$\mathrm{dist}(g^{\varphi_2\circ\psi_{n+1,\infty}}_{x_{n+1,j},s_n(u'_n)^*u'_{n+1}s^*_{n+1}}, \overline{1_A}) < \gamma'_{n+1}, \quad j = 1, ..., m(n+1),$$

which is

$$\mathrm{dist}(g^{\varphi_2\circ\psi_{n+1,\infty}}_{x_{n+1,j},u_n^*u_{n+1}}, \overline{1_A}) < \gamma'_{n+1}, \quad j = 1, ..., m(n+1). \tag{e 4.68}$$

Moreover, it also follows from the definition of Δ_n such that

$$\mu_{\tau\circ\varphi_2\circ\psi_{n,\infty}}(O_a) \geq \Delta_n(a), \quad \forall \tau \in \mathrm{T(A)}, \tag{e 4.69}$$

where O_a is any open ball in X_n with radius $a \geq \eta'_n$.

With (e 4.66), (e 4.67), (e 4.68) and (e 4.69), one applies Theorem 3.9 to obtain a path of unitaries $\{z_n(t) : t \in [0, 1]\}$ in A such that

$$z_n(0) = 0, \quad z_n(1) = u_n^* u_{n+1},$$

and

$$\|[z(t), \varphi_2 \circ \psi_{n+1,\infty}]\| < 1/2^{n+1}, \quad \forall t \in [0, 1].$$

Define

$$u(t + n - 1) = u_n z_{n+1}(t), \quad t \in (0, 1],$$

and then $\{z(t); t \in [0, \infty)\}$ is a continuous path of unitary in A.

Note that

$$\begin{aligned} \mathrm{ad}u(t+n-1) \circ \varphi_1 \quad &\approx_{\delta'_{n+1}} \quad \mathrm{ad}(z_{n+1}(t)) \circ \varphi_2 \\ &\approx_{1/2^{n+1}} \quad \varphi_2 \end{aligned}$$

on $\mathcal{F}_{n+1}$ for all $t \in (0, 1)$. It then follows that

$$\lim_{t\to\infty} u^*(t)\varphi_1(a)u(t) = \varphi_2(a)$$

for all $a \in C$, as desired. □

Let C and A be two unital C*-algebras. Recall that (see10.2 of [9])

$$H_1(K_0(C), K_1(A)) := \{x \in K_1(A) : h([1_C]) = x \text{ for some } h \in \mathrm{Hom}(K_0(C), K_1(A))\}.$$

Lemma 4.3. *Let C be a unital AH-algebra and let A be a unital separable simple C*-algebra with $\mathrm{TR}(A) \le 1$. Suppose that $\varphi, \psi : C \to A$ are two unital monomorphisms. Suppose that $\{\mathcal{F}_n\}$ is an increasing sequence of finite subsets of C such that $\cup_{n=1}^{\infty} \mathcal{F}_n$ is dense in C, and suppose that $\{\mathcal{P}_n\}$ is an increasing sequence of finite subsets of $K_1(C)$ such that its union is $K_1(C)$. Suppose also that there is a sequence of decreasing positive numbers $\delta_n > 0$ with $\sum_{n=1}^{\infty} \delta_n < \infty$ and a sequence of unitaries $\{u_n\} \subset A$ such that*

$$\mathrm{Ad}\, u_n \circ \varphi \approx_{\delta_n} \psi \text{ on } \mathcal{F}_n \text{ and} \tag{e 4.70}$$

$$\rho_A(\mathrm{bott}_1(\psi, u_n^* u_{n+1})(x) = 0 \;\; \textit{for all} \;\; x \in \mathcal{P}_n. \tag{e 4.71}$$

Then we may further require that $u_n \in U_0(A)$ if $H_1(K_0(C), K_1(A)) = K_1(A)$.

Proof. The proof is exactly the same as that of Lemma 10.4 of [12]. Note that, we will apply the second part of 3.16 instead of 6.3 of [12]. □

Theorem 4.4. *Let C be a unital AH-algebra and let A be a unital separable simple C*-algebra with $\mathrm{TR}(A) \le 1$. Suppose that $H_1(K_0(C), K_1(A)) = K_1(A)$ and suppose that $\varphi, \psi : C \to A$ are two unital monomorphisms which are asymptotically unitarily equivalent. Then they are strongly asymptotically unitarily equivalent, i.e., there exists a continuous path of unitaries $\{u(t) : t \in [0, \infty)\} \subset U(A)$ such that*

$$u(0) = 1_A \text{ and } \lim_{t \to \infty} u(t)^* \varphi(c) u(t) = \psi(c) \;\; \textit{for all} \;\; c \in C.$$

Proof. The proof is exactly the same as that of Theorem 10.5 in [12]. However, we apply 4.3 instead of Lemma 10.4 of [12] as needed in the proof of [12]. □

Corollary 4.5. *Let X be a compact metric space and let A be a unital separable simple C*-algebra with $\mathrm{TR}(A) \le 1$. Suppose that $\varphi, \psi : C \to A$ are two unital monomorphisms. Then φ and ψ are strongly asymptotically unitarily equivalent if and only if*

$$[\varphi] = [\psi] \text{ in } KK(C(X), A), \; \varphi^{\ddagger} = \psi^{\ddagger}, \tag{e 4.72}$$

$$\tau \circ \varphi = \tau \circ \psi \;\; \textit{and} \;\; \overline{R_{\varphi,\psi}} = 0. \tag{e 4.73}$$

Proof. Note that $K_0(C(X)) = (\mathbb{Z} \cdot [1_{C(X)}]) \oplus G$ for some abelian subgroup G of $K_0(C(X))$. For each $x \in K_1(A)$, define a homomorphism $h : K_0(C(X)) \to K_1(A)$ by $h([1_{C(X)}]) = x$ and $h|_G = 0$. In other words, one has that $H_1(K_0(C), K_1(A)) = K_1(A)$. □

Proposition 4.6. *Let C be a unital amenable C*-algebra satisfying the UCT and let A be a unital separable simple C*-algebra with $\mathrm{TR}(A) \le 1$. Suppose that φ and ψ are two unital monomorphisms. Suppose also that*

$$[\varphi] = [\psi] \text{ in } KL(C, A), \tag{e 4.74}$$

$$\tau \circ \varphi = \tau \circ \psi \;\; \textit{for all} \;\; \tau \in T(A) \text{ and} \tag{e 4.75}$$

$$R_{\varphi,\psi}(K_1(M_{\varphi,\psi})) \subset \rho_A(K_0(A)). \tag{e 4.76}$$

Then

$$\varphi^{\dagger} = \psi^{\dagger}. \tag{e 4.77}$$

Proof. Let $u \in M_l(C)$ be a unitary, where $l \geq 1$ is an integer. Let $z \in M_l(M_{\varphi,\psi})$ be a unitary which is piecewise smooth on $[0,1]$ such that $\pi_0 \circ z = \varphi(u)$ and $\pi_1 \circ z = \psi(u)$. Let G be a finitely generated subgroup of $K_1(C)$ which contains $[u]$. By the assumption, there is an injective homomorphism $\theta_G : G \to K_1(M_{\varphi,\psi})$ such that

$$(\pi_0)_{*1} \circ \theta_G = \mathrm{id}_G \ \text{ and } \ R_{\varphi,\psi} \circ \theta_G \in \mathrm{Hom}(G, \rho_A(K_0(A))). \tag{e 4.78}$$

It follows that there exists projections $p, q \in M_{l'}(A)$ such that

$$\theta([u]) = [zv] \in K_1(M_{\varphi,\psi}), \tag{e 4.79}$$

where $v(t) = (e^{2\pi i t}p + (1-p))(e^{-2\pi i t}p + (1-p)) \in M_{l'}(M_{\varphi,\psi})$. To simplify the notation, without loss of generality, we may assume that $l = l'$. By (e 4.78),

$$R_{\varphi,\psi}([zv]) \in \rho_A(K_0(A)). \tag{e 4.80}$$

Since $R_{\varphi,\psi}([v]) \in \rho_A(K_0(A))$, one computes that

$$R_{\varphi,\psi}([z]) \in \rho_A(K_0(A)). \tag{e 4.81}$$

Now let $w(t) \in C([0,1], A)$ be a unitary which is piecewise smooth such that $w(0) = \psi(u)^*\varphi(u)$ and $w(1) = 1_{M_l(A)}$. Then

$$\psi(u)w(t) \in M_l(M_{\varphi,\psi}). \tag{e 4.82}$$

Moreover $[z] = [\psi(u)w]$ in $K_1(M_{\varphi,\psi})$. It follows that, for any $\tau \in T(A)$,

$$\begin{aligned}
\frac{1}{2\pi i}\int_0^1 \tau(\frac{d(w(t))}{dt}w^*(t)))dt &= \frac{1}{2\pi i}\int_0^1 \tau(\psi(u)\frac{d(w(t))}{dt}w^*(t))\psi(u)^*)dt && \text{(e 4.83)}\\
&= \frac{1}{2\pi i}\int_0^1 \tau(\frac{d(\psi(u)w(t))}{dt})(\psi(u)w(t)^*)dt && \text{(e 4.84)}\\
&= R_{\varphi,\psi}([z])(\tau). && \text{(e 4.85)}
\end{aligned}$$

Thus, by (e 4.81), there exists $x \in K_0(A)$ such that

$$\mathrm{Det}([w])(\tau) = \rho_A(x)(\tau) \tag{e 4.86}$$

for all $\tau \in T(A)$. It follows from a result of P. Ng ([16]) that

$$\psi(u^*)\varphi(u) \in DU(M_l(A)).$$

Since this holds for all unitaries $u \in M_l(C)$, it follows that

$$\varphi^\dagger = \psi^\dagger.$$

$\square$

Corollary 4.7. *Let C be a unital AH-algebra and let A be a unital separable simple C^*-algebra with $TR(A) \leq 1$. Suppose that $\varphi_1, \varphi_2 : C \to A$ are two unital monomorphisms. Then φ and ψ are asymptotically unitarily equivalent if and only if*

$$\begin{aligned}
[\varphi_1] &= [\varphi_2] \ \textit{in}\ KK(C,A), && \text{(e 4.87)}\\
\tau \circ \varphi &= \tau \circ \psi \ \textit{ for all }\ \tau \in T(A) \ \textit{ and} && \text{(e 4.88)}\\
\overline{R}_{\varphi_1,\varphi_2} &= 0. && \text{(e 4.89)}
\end{aligned}$$

Proof. We only need to show the "if part" of the statement. It follows from 4.6 that, in addition, one has

$$\varphi^{\dagger} = \psi^{\dagger}. \tag{e 4.90}$$

This of course implies that $\varphi^{\ddagger} = \psi^{\ddagger}$. Then 4.2 applies. □

Theorem 4.8. *Let C be a unital AH-algebra and let A be a unital simple C^*-algebra with $TR(A) \le 1$. Suppose that $\varphi, \psi : C \to A$ are two unital monomorphisms such that*

$$[\varphi] = [\psi] \text{ in } KK(C, A), \tag{e 4.91}$$
$$\tau \circ \varphi = \tau \circ \psi, \text{ for all } \tau \in T(A), \text{ and} \tag{e 4.92}$$
$$\varphi^{\dagger} = \psi^{\dagger}, \tag{e 4.93}$$

then φ and ψ are asymptotically unitarily equivalent, provided that one of the following holds:

(1) *$K_1(C)$ is finitely generated, or*

(2) *$K_0(A)$ is finitely generated, or*

(3) *the short exact sequence*

$$0 \to \mathrm{kre}\rho_A \to K_0(A) \to \rho_A(K_0(A)) \to 0$$

splits.

Proof. Let C and A be as in the statement. Suppose that φ, $\psi : C \to A$ are two unital monomorphisms which satisfy the assumptions (e 4.91), (e 4.92) and (e 4.93). In particular, (e 4.93) implies that

$$\varphi^{\ddagger} = \psi^{\ddagger}. \tag{e 4.94}$$

Since $[\varphi] = [\psi]$, there exists a splitting map $\theta : \underline{K}(C) \to \underline{K}(M_{\varphi,\psi})$ such that

$$\theta \circ [\pi_0] = [\mathrm{id}_C]. \tag{e 4.95}$$

Let $u \in M_l(C)$ be a unitary for some integer $l \ge 1$. Let $z \in M_l(M_{\varphi,\psi})$ be a unitary such that $z(0) = \varphi(u)$ and $z(1) = \psi(u)$. Moreover, we may assume that z is piecewise smooth. Define $z_1(t) = \psi(u)^* z(t)$ for $t \in [0, 1]$. Then z_1 is a piecewise smooth and continuous path of unitaries in A such that $z_1(0) = \psi(u)^*\varphi(u)$ and $z_1(1) = 1_{M_l}$. It follows from (e 4.93) that

$$\frac{1}{2\pi i}\int_0^1 \tau(\frac{dz_1(t)}{dt} z_1(t)^*)dt \in \rho_A(K_0(A)), \tag{e 4.96}$$

where $\tau \in T(A)$. One then easily computes that

$$R_{\varphi,\psi}([z]) \in \rho_A(K_0(A)). \tag{e 4.97}$$

On the other hand, there is a projection $p \in M_{l'}(A)$ such that the following holds:

$$\theta([u]) = [zv], \tag{e 4.98}$$

where $v(t) = e^{2\pi i t}p + (1_{M_{l'}} - p)$ for all $t \in [0, 1]$. To simplify the notation, without loss of generality, we may assume that $l' = l$. It follows that

$$R_{\varphi,\psi}([zv]) = R_{\varphi,\psi}([z]) + R_{\varphi,\psi}([v]) \in \rho_A(K_0(A)). \tag{e 4.99}$$

It follows that

$$R_{\varphi,\psi}\circ\theta\in \mathrm{Hom}(K_1(C),\rho_A(K_0(A))). \tag{e 4.100}$$

In all three cases (1), (2) and (3), there exists a homomorphism $\lambda_0 : R_{\varphi,\psi}\circ\theta(K_1(C))\to K_0(A)$ such that

$$\rho_A\circ\lambda_0=\mathrm{id}_{R_{\varphi,\psi}\circ\theta(K_1(C))}. \tag{e 4.101}$$

Define $\lambda=\lambda_0\circ R_{\varphi,\psi}\circ\theta$. So λ is a homomorphism from $K_1(C)$ into $K_0(A)$. Define, by viewing $K_0(A)$ as a subgroup of $K_1(M_{\varphi,\psi})$,

$$\theta_1=\theta-\lambda. \tag{e 4.102}$$

Then

$$R_{\varphi,\psi}\circ\theta_1=0. \tag{e 4.103}$$

It follows that

$$\overline{R_{\varphi,\psi}}=0. \tag{e 4.104}$$

The theorem then follows from 4.2. □

Corollary 4.9. *Let X be a finite CW-complex and let A be a unital simple C*-algebra with finite tracial rank. Suppose that $\varphi,\psi : C(X)\to A$ are two unital monomorphisms. Then φ and ψ are asymptotically unitarily equivalent if and only if*

$$[\varphi]=[\psi]\ \text{ in }\ KK(C,A), \tag{e 4.105}$$
$$\tau\circ\varphi=\tau\circ\psi\ \textit{ for all }\ \tau\in T(A)\ \text{ and} \tag{e 4.106}$$
$$\varphi^{\dagger}=\psi^{\dagger}. \tag{e 4.107}$$

Remark 4.10. We would point out that the assumptions in 4.2 is more sensitive than those in (e 4.91), (e 4.92) and (e 4.93), in general.

Let A be a unital simple AF-algebra with $K_0(A)$ given by a non-splitting short exact sequence

$$0\to G\to K_0(A)\to\mathbb{Q}\to 0, \tag{e 4.108}$$

where G is a countable abelian group and where the order of an element is determined by its image in $\mathbb{Q}$. In particular, A has a unique tracial state τ and $\rho_A(K_0(A))=\mathbb{Q}$. Let C be a unital simple C*-algebra of tracial rank zero with $K_1(C)=\mathbb{Q}\oplus\mathrm{Tor}(K_1(C))$ which also satisfies the UCT. Let $\kappa\in KK(C,A)^{++}$ such that $\kappa([1_C])=[1_A]$. Then there exists a unital monomorphism $\varphi : C\to A$ such that $[\varphi]=\kappa$. Let $\lambda=\varphi_T : T(A)\to T(C)$ be the affine continuous map induced by φ. Let $\gamma : K_1(C)\to\rho_A(K_0(A))$ be an isomorphism as abelian group. It follows from 4.8 of [13] that there exists a unital monomorphism $\psi : C\to A$ such that $[\psi]=\kappa=[\varphi]$, $\psi_T=\lambda=\varphi_T$ and there exists a splitting map $\theta : \underline{K}(C)\to\underline{K}(M_{\varphi,\psi})$ such that

$$R_{\varphi,\psi}\circ\theta=\gamma+\gamma_0, \tag{e 4.109}$$

where $\gamma_0\in\mathcal{R}_0$. We may write $\gamma_0=\rho_A\circ f$, where $f : K_1(C)\to K_0(A)$ is a homomorphism. It follows from 4.6 that

$$\varphi^{\dagger}=\psi^{\dagger}. \tag{e 4.110}$$

However, there is no homomorphism $\lambda_1 : K_1(C) \to K_0(A)$ such that

$$R_{\varphi,\psi} \circ \theta = \rho_A \circ \lambda_1.$$

Otherwise, $\eta = (\lambda_1 - f) \circ \gamma^{-1}$ would split the short exact sequence (e 4.108), since

$$\begin{aligned} \rho_A \circ \eta &= \rho_A \circ (\lambda_1 - f) \circ \gamma_1^{-1} && \text{(e 4.111)} \\ &= (R_{\varphi,\psi} \circ \theta - \rho_A \circ f) \circ \gamma_1^{-1} && \text{(e 4.112)} \\ &= (\gamma + \gamma_0 - \gamma_0) \circ \gamma^{-1} = \mathrm{id}_{\rho_A(K_0(A))}. && \text{(e 4.113)} \end{aligned}$$

In other words,

$$\overline{R_{\varphi,\psi}} \neq 0.$$

References

[1] B. Blackadar, A. Kumjian, and M. Rørdam. Approximately central matrix units and the structure of noncommutative tori. *K-Theory*, 6(3):267–284, 1992.

[2] M. Dădărlat and T. Loring. A universal multicoefficient theorem for the Kasparov groups. *Duke Math. J.*, 84(2):355–377, 1996.

[3] G. Gong. On the classification of simple inductive limit C*-algebras. I. The reduction theorem. *Doc. Math.*, 7:255–461, 2002.

[4] A. Kishimoto and A. Kumjian. The Ext class of an approximately inner automorphism. II. *J. Operator Theory*, 46:99–122, 2001.

[5] Liangqing Li. Simple inductive limit C*-algebras: spectra and approximations by interval algebras. *J. Reine Angew. Math.*, 507:57–79, 1999.

[6] H. Lin. Classification of homomorphisms and dynamical systems. *Trans. Amer. Math. Soc.*, 359(2):859–895, 2007.

[7] H. Lin. Localizing the Elliott conjecture at strongly self-absorbing C*-algebras, II—an appendix. *arXiv: 0709.1654v1*, 2007.

[8] H. Lin. Simple nuclear C*-algebras of tracial topological rank one. *J. Funct. Anal.*, 251(2):601–679, 2007.

[9] H. Lin. Asymptotically unitary equivalence and asymptotically inner automorphisms. *Amer. J. Math.*, 131:1589–1677, 2009.

[10] H. Lin. Approximate homotopy of homomorphisms from C(X) into a simple C*-algebra. *Mem. Amer. Math. Soc.*, 205(963):vi+131, 2010.

[11] H. Lin. Homotopy of unitaries in simple C*-algebras with tracial rank one. *Internat. J. Math.*, 21(10), 2010.

[12] H. Lin. Asymptotically unitary equivalence and classification of simple amenable C*-algebras. *Invent. Math.*, 183(2):385–450, 2011.

[13] H. Lin and Z. Niu. Lifting KK-elements, asymptotic unitary equivalence and classification of simple C*-algebras. *Adv. Math.*, 219:1729–1769, 2008.

[14] H. Lin. Homomorphisms from AH-algebras. *arXiv: 1102.4631v1*, 2011.

[15] H. Lin. On local AH algebras. *arXiv:1104.0445v5*.

[16] P. Ng. The kernel of determinant map on certain simple C*-algebras. *Preprint*, 2012, *arXiv: 1206.6168*.

[17] P. Ng and W. Winter. Commutative C*-subalgebras of simple stably finite C*-algebras with real rank zero. *Indiana Univ. Math. J.*, 57(7):3209–3239, 2008.

[18] N. C. Phillips. How may exponentials. *Amer. J. Math.*, 116(6):1513–1543, 1994.

[19] M. C. Rieffel The homotopy groups of the unitary groups of noncommutative tori. *J. Operator Theory*, 17(2):237–254, 1987.

[20] K. Thomsen. Traces, unitary characters and crossed products by $\mathbb{Z}$. *Publ. Res. Inst. Math. Sci.*, 31(6):1011–1029, 1995.

[21] W. Winter. Localizing the Elliott conjecture at strongly self-absorbing C*-algebras. *ArXiv: 0708.0283v3*, 2007.